< ラプラス変換 2 >

ラプラス変換の性質をいくつか示す。

1　$\mathscr{L}[a_1 f_1(t) + a_2 f_2(t)] = a_1 \mathscr{L}[f_1(t)] + a_2 \mathscr{L}[f_2(t)]$

2　$\mathscr{L}[f(t)] = F(s)$ のとき $\mathscr{L}[f(\alpha t)] = \dfrac{1}{\alpha} F\left(\dfrac{s}{\alpha}\right) \qquad (\alpha > 0)$

3　$\mathscr{L}[f(t)] = F(s)$ のとき $\mathscr{L}[e^{\alpha t} f(t)] = F(s-\alpha)$

(証明)

$$\mathscr{L}[e^{\alpha t} f(t)] = \int_0^\infty e^{\alpha t} f(t) e^{-st} dt = \int_0^\infty f(t) e^{-(s-\alpha)t} dt = F(s-\alpha)$$

4　$\mathscr{L}[f(t)] = F(s)$ のとき $\mathscr{L}[f_\alpha(t)] = e^{-\alpha s} F(s) \qquad (\alpha > 0)$

ここで $f_\alpha(t) = \begin{cases} f(t-\alpha) & : \ t \geqq \alpha \\ 0 & : \ t < \alpha \end{cases}$

(証明)

$$\mathscr{L}[f_\alpha(t)] = \int_\alpha^\infty f(t-\alpha) e^{-st} dt = \int_0^\infty f(\tau) e^{-s(\tau+\alpha)} d\tau \qquad (t-\alpha = \tau)$$

$$= e^{-\alpha s} \int_0^\infty f(\tau) e^{-s\tau} d\tau = e^{-\alpha s} F(s)$$

例 1　$$\mathscr{L}[e^{ikt}] = \int_0^\infty e^{(ik-s)t} dt = \lim_{b\to\infty} \left[\frac{1}{ik-s} e^{(ik-s)t} \right]_{t=0}^{t=b} = \frac{1}{s-ik} \qquad (\mathrm{Re}(s) > 0)$$

例 2　$$\mathscr{L}[\cos(kt)] = \mathscr{L}\left[\frac{e^{ikt}+e^{-ikt}}{2}\right] = \frac{1}{2}\{\mathscr{L}[e^{ikt}] + \mathscr{L}[e^{-ikt}]\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-ik} + \frac{1}{s+ik}\right\} = \frac{s}{s^2+k^2}$$

問　次のラプラス変換を求めよ

(1) $\mathscr{L}[\sin(kt)]$

(2) $\mathscr{L}[e^{\alpha t}\cos(kt)]$

(3) $\mathscr{L}[e^{\alpha t}\sin(kt)]$