< 広義積分 2 >

定理　定数 α, β に対し、次式が成り立つ。ただし $\alpha > 0$。

(1) $\displaystyle\int_0^\infty e^{-\alpha t}\cos(\beta t)dt = \frac{\alpha}{\alpha^2+\beta^2}$

(2) $\displaystyle\int_0^\infty e^{-\alpha t}\ \frac{\sin(\beta t)}{t}dt = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$

(3) $\displaystyle\int_0^\infty \frac{\sin t}{t}dt = \frac{\pi}{2}$

< 証明の概略 >

(1) $\displaystyle I = \int_0^\infty e^{-\alpha t}\cos(\beta t)dt$ とおくと部分積分法より

$$I = \left[-\frac{1}{\alpha}e^{-\alpha t}\cos(\beta t)\right]_{t=0}^{t=\infty} - \int_0^\infty \frac{1}{\alpha}e^{-\alpha t}\beta\sin(\beta t)dt = \frac{1}{\alpha} - \frac{\beta}{\alpha}\int_0^\infty e^{-\alpha t}\sin(\beta t)dt$$

$$= \frac{1}{\alpha} - \frac{\beta}{\alpha}\left\{\left[-\frac{1}{\alpha}e^{-\alpha t}\sin(\beta t)\right]_{t=0}^{t=\infty} + \int_0^\infty \frac{1}{\alpha}e^{-\alpha\beta}\beta\cos(\beta t)dt\right\}$$

$$= \frac{1}{\alpha} - \left(\frac{\beta}{\alpha}\right)^2\int_0^\infty e^{-\alpha t}\cos(\beta t)dt = \frac{1}{\alpha} - \left(\frac{\beta}{\alpha}\right)^2 I$$

であるから

$$I = \frac{1}{\alpha} - \left(\frac{\beta}{\alpha}\right)^2 I \quad \Rightarrow \quad I = \frac{\frac{1}{\alpha}}{1+\left(\frac{\beta}{\alpha}\right)^2} = \frac{\alpha}{\alpha^2+\beta^2}$$

(2) $\displaystyle f_\alpha(x) = \int_0^\infty e^{-\alpha t}\ \frac{\sin(xt)}{t}dt$ とおいて x で微分すると

$$\frac{d}{dx}f_\alpha(x) = \int_0^\infty e^{-\alpha t}\frac{\frac{d}{dx}\sin(xt)}{t}dt = \int_0^\infty e^{-\alpha t}\cos(xt)dt = \frac{\alpha}{\alpha^2+x^2}$$

よって $\displaystyle f_\alpha(x) = \tan^{-1}\left(\frac{x}{\alpha}\right) + C$ (C は定数)。　ここで $x=0$ のとき

$\displaystyle f_\alpha(0) = \int_0^\infty e^{-\alpha t}\frac{0}{t}dt = 0$　　より　　$C=0$　　よって　　$\displaystyle f_\alpha(x) = \tan^{-1}\left(\frac{x}{\alpha}\right)$

従って $\displaystyle\int_0^\infty e^{-\alpha t}\frac{\sin(\beta t)}{t}dt = f_\alpha(\beta) = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$

(3) $\displaystyle\int_0^\infty \frac{\sin t}{t}dt = \lim_{\alpha\to+0}\int_0^\infty e^{-\alpha t}\frac{\sin t}{t}dt = \lim_{\alpha\to+0}\tan^{-1}\left(\frac{1}{\alpha}\right) = \frac{\pi}{2}$