高知工科大学

Kochi University of Technology

数学 5

(2007年度版)

解答

< 1ページ. 平面の方程式 >

問の解答

$(0,0,1)$, $(1,0,0)$, $(0,1,0)$

＜ 2 ページ.2 変数関数 ＞

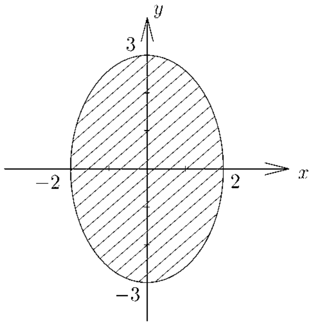
問の解答

(1) 定義域

$$\frac{x^2}{4}+\frac{y^2}{9} \leqq 1$$

値域

$$0 \leqq z \leqq 1$$

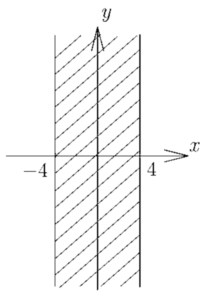


(2) 定義域

$$-4 \leqq x \leqq 4$$

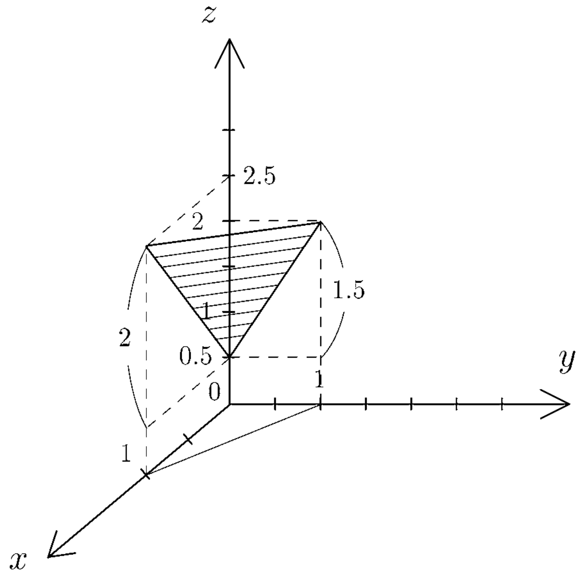
値域

$$-4 \leqq z \leqq 0$$



＜ 3 ページ.2 変数関数のグラフ ＞

問の解答



＜ 5 ページ. 偏微分 2 ＞

問の解答

(1) $f(x,1) = x^3 - 2x^2 + x + 1$ 　　, $f_x(x,1) = 3x^2 - 4x + 1$

$f(x,2) = x^3 - 4x^2 + 4x + 8$ 　　, $f_x(x,2) = 3x^2 - 8x + 4$

$f_x(x,y) = 3x^2 - 4xy + y^2$

(2) $f(x,1) = x^4 - 2x^2 + 3x - 5$ 　　, $f_x(x,1) = 4x^3 - 4x + 3$

$f(x,2) = x^4 - 8x^2 + 24x - 10$ 　　, $f_x(x,2) = 4x^3 - 16x + 24$

$f_x(x,y) = 4x^3 - 4xy^2 + 3y^3$

＜ 6 ページ. 偏微分 3 ＞

問の解答

(1) $f_x(x, y) = 6x - 1 + 2y$

(2) $f_x(x, y) = 5x^4 + 20x^3y - 3x^2y^2 - 4xy^3 + 6y^4$

(3) $e^y + 2y^3e^{2x}$

(4) $2x\cos y + \cos x\cos y$

(5) $f_x(x, y) = \log y - \dfrac{2x}{y}$

＜ 7 ページ. 偏微分 4 ＞

問の解答

(1) $f(1,y) = 1 - 2y + y^2 + y^3$, $f_y(1,y) = -2 + 2y + 3y^2$

$f(2,x) = 8 - 8y + 2y^2 + y^3$, $f_y(2,y) = -8 + 4y + 3y^3$

$f_y(x,y) = -2x^2 + 2xy + 3y^2$

(2) $f(1,y) = 1 - 2y^2 + 3y^3 - 5y$, $f_y(1,y) = -4y + 9y^2 - 5$

$f(2,y) = 16 - 8y^2 + 6y^3 - 5y$, $f_y(2,y) = -16y + 18y^2 - 5$

$f_y(x,y) = -4x^2y + 9xy^2 - 5$

＜ 8 ページ. 偏関数 5 ＞

問の解答

(1) $f_y(x,y) = 2x + 10y - 6$

(2) $f_y(x,y) = 5x^4 - 2x^3y - 6x^2y^2 + 24xy^3 + 42y^5$

(3) $f_y(x,y) = xe^y + 3y^2e^{2x}$

(4) $-x^2\sin y - \sin x \sin y$

(5) $\dfrac{x}{y} + \dfrac{x^2}{y^2}$

＜ 9 ページ. 偏微分 6 ＞

問の解答

(1) $\dfrac{\partial}{\partial x}(x^3 - x^2y^2 + 3xy^5)$, $\dfrac{\partial}{\partial y}(x^3 - x^2y^2 + 3xy^5)$

$= 3x^2 - 2xy^2 + 3y^5$ $= -2x^2y + 15xy^4$

(2) $\dfrac{\partial}{\partial x}(e^x \cos y) = e^x \cos y$, $\dfrac{\partial}{\partial y}(e^x \cos y) = -e^x \sin y$

(3) $\dfrac{\partial}{\partial x}\left(\dfrac{\log y}{x}\right) = -\dfrac{\log y}{x^2}$, $\dfrac{\partial}{\partial y}\left(\dfrac{\log y}{x}\right) = \dfrac{1}{xy}$

< 10 ページ. 偏微分 7 >

問の解答

(1) $\dfrac{\partial z}{\partial x} = 5\left(2x+y^2\right)^4 \times \dfrac{\partial}{\partial x}\left(2x+y^2\right)$ $= 10\left(2x+y^2\right)^4$

, $\dfrac{\partial z}{\partial y} = 5\left(2x+y^2\right)^4 \times \dfrac{\partial}{\partial y}\left(2x+y^2\right)$ $= 10y\left(2x+y^2\right)^4$

(2) $\dfrac{\partial z}{\partial x} = \dfrac{1}{2\sqrt{1-2x+3y^2}} \times \dfrac{\partial}{\partial x}\left(1-2x+3y^2\right)$ $= -\dfrac{1}{\sqrt{1-2x+3y^2}}$

, $\dfrac{\partial z}{\partial y} = \dfrac{1}{2\sqrt{1-2x+3y^2}} \times \dfrac{\partial}{\partial y}\left(1-2x+3y^2\right)$ $= \dfrac{3y}{\sqrt{1-2x+3y^2}}$

(3) $\dfrac{\partial z}{\partial x} = 3e^{3x-y^2}$, $\dfrac{\partial z}{\partial y} = -2ye^{3x-y^2}$

(4) $\dfrac{\partial z}{\partial x} = \dfrac{-\cos x\cos y}{1-\sin x\cos y}$, $\dfrac{\partial z}{\partial y} = \dfrac{\sin x\sin y}{1-\sin x\cos y}$

＜ 11 ページ. 偏微分 8 ＞

問の解答

(1) $f_x(x,y) = 5x^4 - 4x^3y + 4xy^3$, $f_y(x,y) = -x^4 + 6x^2y^2 - 28y^3$

(2) $f_x(x,y) = -5\sin\left(5x - y^2\right)$, $f_y(x,y) = 2y\sin\left(5x - y^2\right)$

(3) $\dfrac{\partial z}{\partial x} = -\dfrac{y}{(xy - 2y^3)^2}$, $\dfrac{\partial z}{\partial y} = -\dfrac{x - 6y^2}{(xy - 2y^3)^2}$

(4) $z_x = \dfrac{y}{2\sqrt{xy + y^2}}$, $z_y = \dfrac{x + 2y}{2\sqrt{xy + y^2}}$

(5) $z_x = -y^2 e^{3y - xy^2 + y^3}$, $z_y = \left(3 - 2xy + 3y^2\right) e^{3y - xy^2 + y^3}$

< 12 ページ.2 階偏導関数 1 >

問の解答

(1) $f_x(x,y) = 5x^4 - 8x^3y + 6xy^2$, $f_y(x,y) = -2x^4 + 6x^2y + 16y^3$

$f_{xx}(x,y) = 20x^3 - 24x^2y + 6y^2$, $f_{yy}(x,y) = 6x^2 + 48y^2$

(2) $\dfrac{\partial z}{\partial x} = -2\sin(2x)\sin\left(y^2\right)$, $\dfrac{\partial z}{\partial y} = 2y\cos(2x)\cos\left(y^2\right)$

$\dfrac{\partial^2 z}{\partial x^2} = -4\cos(2x)\sin\left(y^2\right)$, $\dfrac{\partial^2 z}{\partial y^2} = 2\cos(2x)\cos\left(y^2\right) - 4y^2\cos(2x)\sin\left(y^2\right)$

＜ 13 ページ.2 階編導関数 2 ＞

問の解答

(1) $f_x(x,y) = 5x^4 - 8x^3y + 6xy^2$ 　, $f_y(x,y) = -2x^4 + 6x^2y + 16y^3$

$f_{xy}(x,y) = -8x^3 + 12xy$ 　, $f_{yx}(x,y) = -8x^3 + 12xy$

(2) $\dfrac{\partial z}{\partial x} = -2\sin(2x)\sin\left(y^2\right)$ 　, $\dfrac{\partial z}{\partial y} = 2y\cos(2x)\cos\left(y^2\right)$

$\dfrac{\partial^2 z}{\partial y\partial x} = -4y\sin(2x)\cos\left(y^2\right)$ 　, $\dfrac{\partial^2 z}{\partial x\partial y} = -4y\sin(2x)\cos\left(y^2\right)$

＜ 14 ページ. 偏微分係数 ＞

問の解答

(1) $f_x(x,y) = 3x^2 - 4xy + 5y^2$　　$, f_y(x,y) = -2x^2 + 10xy - 12y^3$

$f_x(2,1) = 12 - 8 + 5 = 9$　　$, f_y(2,1) = -8 + 20 - 12 = 0$

(2) $f_y(x,y) = -\sin x \sin(2y)$　　$, f_y(x,y) = 2\cos x \cos(2y)$

$f_y\left(\frac{\pi}{3}, \frac{\pi}{4}\right) = -\frac{\sqrt{3}}{2} \times 1 = -\frac{\sqrt{3}}{2}$　　$, f_y\left(\frac{\pi}{3}, \frac{\pi}{4}\right) = 2 \times \frac{1}{2} \times 0 = 0$

(3) $f_x(x,y) = 2\log\left(y^3\right)$　　$, f_y(x,y) = \frac{6x}{y}$

$f_x(1,1) = 0$　　$, f_y(1,1) = 6$

< 15 ページ.2 面の共通部分としての線 >

問 1 の解答

$\underline{\ell_2\ :\ \ x=2,\ z=0.3y+1.4}$

$\underline{L_0\ :\ \ y=0,\ z=0.4x+0.6}$

$\underline{L_1\ :\ \ y=1,\ z=0.4x+0.9}$

問 2 の解答

$\underline{\ell_2\ :\ \ x=2,\ \ z=2+2y-y^2}$

$\underline{L_1\ :\ \ y=1,\ \ z=3}$

$\underline{L_2\ :\ \ y=2,\ \ z=x}$

< 16 ページ. 偏微分係数の幾何学的意味 >

問の解答

$f_x(x, y) = 2x - 3 + y$　　　　$f_y(x, y) = x - 2y + 2$

$f(3, 2) = 9 - 9 + 6 - 4 + 4 - 6 = 0$　　　　$f_x(3, 2) = 6 - 3 + 2 = 5$

$f_y(3, 2) = 3 - 4 + 2 = 1$

接線 L の方程式

$y = 2$

$z = 5(x - 3) + 0$

$\underline{z = 5x - 15}$

接線 ℓ の方程式

$x = 3$

$z = 1(y - 2) + 0$

$\underline{z = y - 2}$

＜ 18 ページ. 接平面 ＞

問の解答

(1) $f_x(x,y) = 2x - y,\ \ f_y(x,y) = -x,\ \ f(3,1) = 9 - 3 = 6$

$f_x(3,1) = 6 - 1 = 5,\ \ f_y(3,1) = -3$

$z = 5(x-3) - 3(y-1) + 6$

$\quad = 5x - 15 - 3y + 3 + 6$ 　　　　(答) $z = 5x - 3y - 6$

(2) $f_x(x,y) = 2x - y,\ \ f_y(x,y) = -x + 4y,\ \ f(1,2) = 1 - 2 + 8 = 7$

$f_x(1,2) = 2 - 2 = 0,\ \ f_y(1,2) = -1 + 8 = 7$

$z = 0(x-1) + 7(y-2) + 7$ 　　　　(答) $z = 7y - 7$

< 20 ページ. 合成関数の微分法 2 >

問の解答

(1) $\dfrac{d}{dt}f\left(2-3t,\ 4-5t\right)=-3f_x\left(2-3t,\ 4-5t\right)-5f_y\left(2-3t,\ 4-5t\right)$

(2) $\dfrac{d}{dt}f\left(r\cos\theta,\ r\sin\theta\right)=\cos\theta\cdot f_x\left(r\cos\theta,\ r\sin\theta\right)+\sin\theta\cdot f_y\left(r\cos\theta,\ r\sin\theta\right)$

＜ 21 ページ. 合成関数の微分法 3 ＞

問の解答

(1) $$\frac{dz}{dt} = 4f_x\left(a+4t,\ b+5t\right) + 5f_y\left(a+4t,\ b+5t\right)$$

$$\begin{aligned}\frac{d^2z}{dt^2} &= 4\left\{4f_{xx}\left(a+4t,\ b+5t\right) + 5f_{xy}\left(a+4t,\ b+5t\right)\right\}\\ &\quad +5\left\{4f_{yx}\left(a+4t,\ b+5t\right) + 5f_{yy}\left(a+4t,\ b+5t\right)\right\}\\ &= 16f_{xx}\left(a+4t,\ b+5t\right) + 40f_{xy}\left(a+4t,\ b+5t\right)\\ &\quad +25f_{yy}\left(a+4t,\ b+5t\right)\end{aligned}$$

(2) $$\frac{dz}{dt} = hf_x\left(a+ht,\ b+kt\right) + kf_y\left(a+ht,\ b+kt\right)$$

$$\begin{aligned}\frac{d^2z}{dt^2} &= h\left\{hf_{xx}\left(a+ht,\ b+kt\right) + kf_{xy}\left(a+ht,\ b+kt\right)\right\}\\ &\quad +k\left\{hf_{yx}\left(a+ht,\ b+kt\right) + kf_{yy}\left(a+ht,\ b+kt\right)\right\}\\ &= h^2f_{xx}\left(a+ht,\ b+kt\right) + 2hkf_{xy}\left(a+ht,\ b+kt\right)\\ &\quad +k^2f_{yy}\left(a+ht,\ b+kt\right)\end{aligned}$$

< 22 ページ. 合成関数の微分法 4 >

問 1 の解答

$$\frac{\partial z}{\partial r} = f_x\left(r\cos\theta,\ r\sin\theta\right)\cos\theta + f_y\left(r\cos\theta,\ r\sin\theta\right)\sin\theta$$

問 2 の解答

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial v}$$

$$= f_x\left(\varphi\left(u, v\right),\ \psi\left(u, v\right)\right)\varphi_v\left(u, v\right)$$

$$+ f_y\left(\varphi\left(u, v\right), \psi\left(u, v\right)\right)\psi_v\left(u, v\right)$$

＜ 24 ページ. ヘシアン ＞

問の解答

(1) $f_x(x,y) = 2x - 3y^2 \quad , \quad f_y(x,y) = -6xy + 4y^3$

$f_{xx}(x,y) = 2 \quad , \quad f_{xy}(x,y) = -6y \quad , \quad f_{yy}(x,y) = -6x + 12y^2$

$f_{xx}(2,1) = 2 \quad , \quad f_{xy}(2,1) = -6 \quad , \quad f_{yy}(2,1) = -12 + 12 = 0$

$H = f_{xx} \times f_{yy} - (f_{xy})^2 = 2 \times 0 - (-6)^2 = -36$

(2) $f_x(x,y) = 2xy^2 - 3 \quad , \quad f_y(x,y) = 2x^2y + 4$

$f_{xx}(x,y) = 2y^2 \quad , \quad f_{xy}(x,y) = 4xy \quad , \quad f_{yy}(x,y) = 2x^2$

$f_{xx}(2,1) = 2 \quad , \quad f_{xy}(2,1) = 8 \quad , \quad f_{yy}(2,1) = 8$

$H = f_{xx} \times f_{yy} - (f_{xy})^2 = 2 \times 8 - 8^2 = 16 - 64 = -48$

$<$ 25ページ.2変数関数の極大・極小 1 $>$

問 1 の解答

$$f_x(0,\ 0) = 0\ ,\ f_y(0,\ 0) = 0\quad f_{xx}(0,\ 0) = -2$$

$$H = f_{xx}(0,\ 0) \times f_{yy}(0,\ 0) - \{f_{xy}(0,\ 0)\}^2 = 4$$

問 2 の解答

$$f_x(0,\ 0) = 0\ ,\ f_y(0,\ 0) = 0\ ,\ f_{xx}(0,\ 0) = 2$$

$$H = f_{xx}(0,\ 0) \times f_{yy}(0,\ 0) - \{f_{xy}(0,\ 0)\}^2 = 2 \times 2 - 1^2 = 3$$

問 3 の解答

$$f_x(0,\ 0) = 0\ ,\ f_y(0,\ 0) = 0\ ,\ f_{xx}(0,\ 0) = -\frac{1}{2}$$

$$H = f_{xx}(0,\ 0) \times f_{yy}(0,\ 0) - \{f_{xy}(0,\ 0)\}^2 = -\frac{1}{4}$$

< 26 ページ.2 変数関数の極大・極小 2 >

問の解答

$$AC - B^2 = f_{xx} \times f_{yy} - (f_{xy})^2 = H\ ,\ A = f_{xx}(a,b)$$

(1)　$f_x(a,b) = f_y(a,b) = 0$ で $H > 0$ かつ $f_{xx}(a,b) > 0$ のとき極小

(2)　$f_x(a,b) = f_y(a,b) = 0$ で $H > 0$ かつ $f_{xx}(a,b) < 0$ のとき極大

< 27 ページ.2 変数関数の極大・極小 3 >

問の解答

$f_x(x,y) = -4x + 2y + 4$, $f_y(x,y) = 2x - 2y - 2$

$f_{xx}(x,y) = -4$, $f_{xy}(x,y) = 2$, $f_{yy}(x,y) = -2$

$f_x(a,b) = -4a + 2b + 4 = 0 \ \Rightarrow\ -2a + b + 2 = 0$

$f_y(a,b) = 2a - 2b - 2 = 0 \ \Rightarrow\ a - b - 1 = 0$

$\underline{a = 1\ ,\ b = 0}$

$H = f_{xx} \times f_{yy} - (f_{xy})^2 = (-4) \times (-2) - 2^2 = 8 - 4 = 4 > 0$

$f_{xx}(1,0) = -4 < 0$ より極大

<u>(答) $(x,y) = (1,0)$ のとき極大値 $f(1,0) = 2$ をとる</u>

< 28ページ.2変数関数の極大・極小 4 >

問の解答

$f_x(x,y) = -6x^2 - 6x$, $f_y(x,y) = -2y$

$f_{xx}(x,y) = -12x - 6$, $f_{xy} = 0$, $f_{yy} = -2$

$f_x = f_y = 0 \ \Rightarrow\ -6x^2 - 6x = 0 \qquad x^2 + x = 0 \ \Rightarrow\ \underline{x = 0, 1}$

$-2y = 0$, $\underline{y = 0}$

① $(x,y) = (0,0)$ のとき

$H = (-6) \times (-2) - 0^2 = 12 > 0$, $f_{xx}(0,0) = -6 < 0$ より極大

② $(x,y) = (-1,0)$ のとき

$H = 6 \times (-2) - 0^2 = -12 < 0$ より極値をとらない

(答) $(x,y) = (0,0)$ のとき極大値 $f(0,0) = 0$ をとる

＜ 29 ページ.2 変数関数の最大・最小 1 ＞

問の解答

$f_x(x,y) = 2x - 2y + 4$, $f_y(x,y) = 4y - 2x - 10$

$f_{xx}(x,y) = 2$, $f_{xy}(x,y) = -2$, $f_{yy}(x,y) = 4$

$$f_x = f_y = 0 \ , \begin{cases} 2x - 2y + 4 = 0 \\ 4y - 2x - 10 = 0 \end{cases} \Rightarrow \ x = 1, \ y = 3$$

$(x,y) = (1,3)$

$H = 2 \times 4 - (-2)^2 = 8 - 4 = 4 > 0$, $f_{xx}(1,3) = 2 > 0$ より極小

$f(1,3) = 1 + 18 - 6 + 4 - 30 = -13$

(答) $(x,y) = (1,3)$ のとき最小値 $f(1,3) = -13$

＜ 30 ページ.2 変数関数の最大・最小 2 ＞

問の解答

$$\mathrm{OP}^2 = x^2 + y^2 + z^2 = x^2 + y^2 + (x + y + 1)^2$$

$$= 2x^2 + 2xy + 2y^2 + 2x + 2y + 1 = f\,(x, y) \text{ とおくと}$$

$$f_x\,(x, y) = 4x + 2y + 2 \text{ , } f_y\,(x, y) = 2x + 4y + 2$$

$$f_{xx} = 4 \text{ , } f_{xy} = 2 \text{ , } f_{yy} = 4 \text{ , } f_x = f_y = 0 \Rightarrow \begin{cases} 4x + 2y + 2 = 0 \\ x + 2y + 1 = 0 \end{cases}$$

$$\Rightarrow x = y = -\frac{1}{3}$$

$$H = 4^2 - 2^2 = 12$$

$$(x, y) = \left(-\frac{1}{3}, -\frac{1}{3}\right) \text{ のとき } z = -\frac{1}{3} - \frac{1}{3} + 1 = \frac{1}{3}$$

$$\ell = \sqrt{\mathrm{OP}^2} = \sqrt{x^2 + y^2 + z^2} = \sqrt{\left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = \sqrt{\frac{3}{9}} = \frac{\sqrt{3}}{3}$$

(答) $\mathrm{P}\left(-\frac{1}{3}\text{ , }-\frac{1}{3}\text{ , }\frac{1}{3}\right)$ のとき最小距離 $l = \frac{\sqrt{3}}{3}$ となる

< 31 ページ. 体積 1 >

問 1 の解答

$$S_1 = \int_{-1}^{1} \Big(\big(4 - x^2\big) - \big(1 - x^2\big) \Big) dx = \int_{-1}^{1} 3dx = 6$$

$$S_2 = 3 \times 2 = 6$$

問 2 の解答

(1) $S(x) = S$

(2) $V = Sh$

< 32 ページ. 体積 2 >

問の解答

(1) $\mathrm{A'C'} = \dfrac{a}{h}x$, $\mathrm{B'C'} = \dfrac{b}{h}x$

(2) $S(x) = \dfrac{1}{2} \times \dfrac{a}{h}x \times \dfrac{b}{h}x = \dfrac{ab}{2h^2}x^2$

(3)
$$V = \int_0^h S(x)\,dx = \int_0^h \frac{ab}{2h^2}x^2 dx$$
$$= \frac{ab}{2h^2}\left[\frac{x^3}{3}\right]_0^h = \frac{ab}{2h^2} \times \frac{h^3}{3} = \frac{abh}{6}$$

< 33 ページ. 体積 3 >

問 1 の解答

$$a = \frac{1}{3}x + 1$$

問 2 の解答

$$b = \frac{1}{3}x + 2$$

問 3 の解答

$$\begin{aligned} S(x) &= \frac{a+b}{2} \times 4 \\ &= \frac{4}{3}x + 6 \end{aligned}$$

問 4 の解答

$$\begin{aligned} V &= \int_0^3 S(x)\,dx \\ &= \int_0^3 \left(\frac{4}{3}x + 6\right) dx \\ &= \left[\frac{2}{3}x^2 + 6x\right]_0^3 \\ &= \frac{2}{3} \times 9 + 6 \times 3 = 24 \end{aligned}$$

< 34 ページ. 体積 4 >

問の解答

$$S(x) = \int_0^3 (5 - x + 0.2y)\,dy$$
$$= \left[5y - xy + 0.1y^2\right]_{y=0}^{y=3}$$
$$= 15 - 3x + 0.9 = 15.9 - 3x$$

$$V = \int_0^2 S(x)\,dx = \int_0^2 (15.9 - 3x)\,dx$$
$$= \left[15.9x - \frac{3}{2}x^2\right]_0^2$$
$$= 31.8 - 6 = 25.8$$

＜ 35 ページ. 体積 5 ＞

問の解答

$$S(x) = \int_0^3 \left(3 - \frac{x^2}{2} + \frac{xy}{2} + 2y - y^2\right) dy$$

$$= \left[3y - \frac{x^2}{2}y + \frac{xy^2}{4} + y^2 - \frac{y^3}{3}\right]_{y=0}^{y=3}$$

$$= 9 - \frac{3}{2}x^2 + \frac{9}{4}x + 9 - 9 = -\frac{3}{2}x^2 + \frac{9}{4}x + 9$$

$$V = \int_0^2 S(x)\,dx = \int_0^2 \left(-\frac{3}{2}x^2 + \frac{9}{4}x + 9\right) dx$$

$$= \left[-\frac{x^3}{2} + \frac{9x^2}{8} + 9x\right]_0^2 = -4 + \frac{9}{2} + 18 = \frac{37}{2}$$

< 36 ページ. 体積 6 >

問の解答

$$S(y) = \int_0^3 \left(\frac{1}{3}x - \frac{1}{4}y + 2\right) dx$$

$$= \left[\frac{x^2}{6} - \frac{y}{4}x + 2x\right]_{x=0}^{x=3} = \frac{9}{6} - \frac{3}{4}y + 6 = -\frac{3}{4}y + \frac{15}{2}$$

$$V = \int_0^4 S(y)\, dy = \int_0^4 \left(-\frac{3}{4}y + \frac{15}{2}\right) dy$$

$$= \left[-\frac{3}{8}y^2 + \frac{15}{2}y\right]_0^4 = -\frac{3}{8} \times 16 + \frac{15}{2} \times 4 = 24$$

＜ 37 ページ. 累次積分 1 ＞

問の解答

$$\int_1^2 \left\{ \int_1^3 \left(x^2 - xy + 1\right) dy \right\} dx = \int_1^2 \left\{ \left[x^2 y - \frac{x}{2} y^2 + y \right]_{y=1}^{y=3} \right\} dx$$

$$= \int_1^2 \left\{ \left(3x^2 - \frac{9}{2}x + 3\right) - \left(x^2 - \frac{x}{2} + 1\right) \right\} dx$$

$$= \int_1^2 \left(2x^2 - 4x + 2\right) dx = \left[\frac{2}{3}x^3 - 2x^2 + 2x \right]_1^2$$

$$= \left(\frac{16}{3} - 8 + 4 \right) - \left(\frac{2}{3} - 2 + 2 \right) = \frac{2}{3}$$

＜ 38 ページ. 累次積分 2 ＞

問の解答

$$\begin{aligned}\int_1^3 \left\{\int_1^2 \left(x^2 - xy + 1\right) dx\right\} dy &= \int_1^3 \left\{\left[\frac{x^3}{3} - \frac{x^2}{2}y + x\right]_{x=1}^{x=2}\right\} dy \\ &= \int_1^3 \left\{\frac{10}{3} - \frac{3}{2}y\right\} dy \\ &= \left[\frac{10}{3}y - \frac{3}{4}y^2\right]_1^3 \\ &= \frac{2}{3}\end{aligned}$$

< 40 ページ. 長方形領域の 2 重積分 2 >

問の解答

$$\iint_D \left(2x - 3y^2\right) dxdy = \int_{-1}^{1} \left\{ \int_0^2 \left(2x - 3y^2\right) dy \right\} dx$$

$$= \int_{-1}^{1} \left\{ \left[2xy - y^3\right]_{y=0}^{y=2} \right\} dx = \int_{-1}^{1} (4x - 8)\, dx$$

$$= \left[2x^2 - 8x\right]_{-1}^{1} = -16$$

＜ 41 ページ. 長方形領域の 2 重積分 3 ＞

問の解答

(1) $D = \left\{(x, y) : 0 \leqq x \leqq 2\ ,\ 0 \leqq y \leqq \dfrac{\pi}{2}\right\}$

$$\iint_D x^3 \sin(2y)\, dxdy = \int_0^2 x^3 dx \times \int_0^{\frac{\pi}{2}} \sin(2y)\, dy$$
$$= \left[\frac{x^4}{4}\right]_{x=0}^{x=2} \times \left[-\frac{\cos(2y)}{2}\right]_{y=0}^{y=\frac{\pi}{2}}$$
$$= \frac{16}{4} \times \left(-\frac{-1}{2} - \left(-\frac{1}{2}\right)\right) = 4$$

(2) $D = \{(x, y) : 0 \leqq x \leqq 1\ ,\ 0 \leqq y \leqq 1\}$

$$\iint_D e^{2x-y} dxdy = \int_0^1 e^{2x} dx \times \int_0^1 e^{-y} dy$$
$$= \left[\frac{1}{2} e^{2x}\right]_{x=0}^{x=1} \times \left[-e^{-y}\right]_{y=0}^{y=1}$$
$$= \frac{1}{2}\left(e^2 - 1\right) \times \left(-e^{-1} + 1\right) = \frac{1}{2}\left(e^2 - 1\right)\left(1 - \frac{1}{e}\right)$$

< 42 ページ. 一般領域の 2 重積分 1 >

問の解答

$$
\begin{aligned}
\iint_D (x+y)\,dxdy &= \int_1^2 \left\{ \int_0^1 (x+y)\,dy \right\} dx + \int_2^3 \left\{ \int_1^2 (x+y)\,dy \right\} dx \\
&= \int_1^2 \left\{ \left[xy + \frac{1}{2}y^2 \right]_{y=0}^{y=1} \right\} dx + \int_2^3 \left\{ \left[xy + \frac{1}{2}y^2 \right]_{y=1}^{y=2} \right\} dx \\
&= \int_1^2 \left(x + \frac{1}{2} \right) dx + \int_2^3 \left(x + \frac{3}{2} \right) dx \\
&= \left[\frac{x^2}{2} + \frac{x}{2} \right]_1^2 + \left[\frac{x^2}{2} + \frac{3}{2}x \right]_2^3 \\
&= 6
\end{aligned}
$$

＜ 43 ページ. 一般領域の 2 重積分 2 ＞

問の解答

$D = \{(x, y) : 0 \leqq x \leqq 1\ ,\ 0 \leqq y \leqq -x + 1\}$ より

$$\begin{aligned}
\iint_D (2x + y)\, dxdy &= \int_0^1 \left\{ \int_0^{-x+1} (2x + y)\, dy \right\} dx \\
&= \int_0^1 \left\{ \left[2xy + \frac{y^2}{2} \right]_{y=0}^{y=-x+1} \right\} dx \\
&= \int_0^1 \left\{ 2x(-x + 1) + \frac{(-x + 1)^2}{2} \right\} dx \\
&= \int_0^1 \left(-\frac{3}{2}x^2 + x + \frac{1}{2} \right) dx \\
&= \left[-\frac{x^3}{2} + \frac{x^2}{2} + \frac{1}{2}x \right]_0^1 \\
&= \frac{1}{2}
\end{aligned}$$

＜ 44 ページ. 一般領域の 2 重積分 3 ＞

問の解答

$D = \{(x, y) : 0 \leqq x \leqq y\ ,\ 0 \leqq y \leqq 1\}$ より

$$\iint_D (xy - y)\,dxdy = \int_0^1 \left\{ \int_0^y (xy - y)\,dx \right\} dy$$
$$= \int_0^1 \left\{ \left[\frac{x^2}{2} y - xy \right]_{x=0}^{x=y} \right\} dy$$
$$= \int_0^1 \left\{ \frac{y^3}{2} - y^2 \right\} dy$$
$$= \left[\frac{y^4}{8} - \frac{y^3}{3} \right]_0^1$$
$$= -\frac{5}{24}$$

< 45 ページ. 変数変換における面積比 >

問 1 の解答

$$S = r_1 r_2 \sin(\beta - \alpha)$$
$$= r_1 r_2 \{\sin\beta\cos\alpha - \cos\beta\sin\alpha\}$$
$$= r_1\cos\alpha\ r_2\sin\beta - r_1\sin\alpha\ r_2\cos\beta$$
$$= a_1 b_2 - a_2 b_1$$

問 2 の解答

$$\frac{\Delta(x,y)}{\Delta(u,v)} = \frac{(a_1\Delta u)(b_2\Delta v) - (a_2\Delta u)(b_1\Delta v)}{\Delta u \times \Delta v}$$
$$= a_1 b_2 - a_2 b_1$$

問 3 の解答

(1) $$\frac{\Delta(x,y)}{\Delta(r,\theta)} = \frac{1}{\Delta r\Delta\theta}\left\{\frac{1}{2}(\Delta\theta)(r+\Delta r)^2 - \frac{1}{2}(\Delta\theta)r^2\right\}$$
$$= \frac{r^2 + 2\Delta r\cdot r + (\Delta r)^2 - r^2}{2\Delta r}$$
$$= r + \frac{1}{2}\Delta r$$

(2) $$\lim_{\substack{\Delta r\to 0\\ \Delta\theta\to 0}} \frac{\Delta(x,y)}{\Delta(r,\theta)} = \lim_{\substack{\Delta r\to 0\\ \Delta\theta\to 0}} \left(r + \frac{1}{2}\Delta r\right) = r$$

< 46ページ. ヤコビアン >

問の解答

(1) $\dfrac{\partial(x,\ y)}{\partial(u,\ v)} = \dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial x}{\partial v}\dfrac{\partial y}{\partial u} = a_1 b_2 - a_2 b_1$

(2) $$\begin{aligned}\frac{\partial(x,\ y)}{\partial(r,\ \theta)} &= \frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r} \\ &= \cos\theta \times r\cos\theta - (-r\sin\theta) \times \sin\theta \\ &= r\left(\cos^2\theta + \sin^2\theta\right) = r\end{aligned}$$

< 48 ページ. 重積分の変数変換 2 >

問の解答

$D = \left\{(x, y) : x^2 + y^2 \leqq R^2,\ x \geqq 0,\ y \geqq 0\right\}$ より

$\Omega = \left\{(r, \theta) : 0 \leqq r \leqq R,\ 0 \leqq \theta \leqq \dfrac{\pi}{2}\right\}$, $x = r\cos\theta,\ y = r\sin\theta$ とおくと

$$
\begin{aligned}
\iint_D e^{-x^2-y^2}\,dxdy &= \iint_\Omega e^{-(r\cos\theta)^2-(r\sin\theta)^2}\left|\frac{\partial(x, y)}{\partial(r, \theta)}\right| drd\theta \\
&= \int_0^{\frac{\pi}{2}}\left\{\int_0^R e^{-r^2} rdr\right\} d\theta \\
&= \int_0^{\frac{\pi}{0}}\left(\left[-\frac{1}{2}e^{-r^2}\right]_{r=0}^{r=R}\right) d\theta \\
&= \int_0^{\frac{\pi}{2}} d\theta \left\{-\frac{1}{2}e^{-R^2} + \frac{1}{2}\right\} \\
&= \left(\frac{1 - e^{-R^2}}{2}\right) \times \frac{\pi}{2}
\end{aligned}
$$

< 49 ページ. 重積分の変数変換 3 >

問 1 の解答

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad , \quad \Omega = \left\{(r,\theta) : 0 \leqq r \leqq R,\ 0 \leqq \theta \leqq \frac{\pi}{2}\right\} \text{ とおくと}$$

$$\begin{aligned} \iint_D \sqrt{R^2 - x^2 - y^2}\,dxdy &= \iint_\Omega \sqrt{R^2 - (r\cos\theta)^2 - (r\sin\theta)^2} \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| drd\theta \\ &= \int_0^{\frac{\pi}{2}} \left\{\int_0^R \sqrt{R^2 - r^2}\,rdr\right\} d\theta \\ &= \int_0^{\frac{\pi}{2}} d\theta \times \int_0^R \left(R^2 - r^2\right)^{\frac{1}{2}} rdr = (*) \text{ とおく} \end{aligned}$$

ここで $u = r^2$ とおくと

$$(*) = [\theta]_0^{\frac{\pi}{2}} \times \frac{1}{2}\int_0^{R^2} \left(R^2 - u\right)^{\frac{1}{2}} \frac{1}{2}du$$

さらに $v = -u$ とおくと

$$\begin{aligned} (*) &= \frac{\pi}{2} \times \frac{1}{2}\int_0^{-R^2} \left(R^2 + v\right)^{\frac{1}{2}} (-1)\,dv \\ &= \frac{\pi}{2} \times \left\{-\frac{1}{2}\left[\frac{2}{3}\left(R^2 + v\right)^{\frac{3}{2}}\right]_{v=0}^{v=-R^2}\right\} \\ &= \frac{\pi}{2} \times \left\{-\frac{1}{2}\left(\frac{2}{3} \times 0 - \frac{2}{3}\left(R^2\right)^{\frac{3}{2}}\right)\right\} \\ &= \frac{\pi}{2} \times \frac{1}{3}R^3 = \frac{\pi}{6}R^3 \end{aligned}$$

問 2 の解答

$$\begin{aligned} V &= 8\iint_D \sqrt{R^2 - x^2 - y^2}\,dxdy \\ &= 8 \times \frac{\pi}{6}R^3 = \frac{4\pi}{3}R^3 \end{aligned}$$