高知工科大学
Kochi University of Technology

基礎数学 ワークブック *No.* 8

「フーリエ解析」

解答

＜１ページ. 三角関数＞

問の解答

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

< 2ページ. 正弦波のグラフ >

問の解答

(1) 周期 2π ， 振幅 $\sqrt{2}$， 初期位相 $-\frac{1}{4}$

$y = \sqrt{2}\sin(t + \frac{\pi}{4})$

(2) 周期 $\frac{2}{3}\pi$ ， 振幅 2， 初期位相 0

$y = 2\sin(3t)$

< 3ページ. 同周期正弦波の和 >

問の解答

(1) $\sin t + \cos t = \sqrt{2}\sin(t + \frac{\pi}{4})$

周期 2π

振幅 $\sqrt{2}$

初期位相 $-\frac{\pi}{4}$

(2) $\sqrt{3}\sin(2t) + \cos(2t) = 2\sin(2t + \frac{\pi}{6})$

周期 π

振幅 2

初期位相 $-\frac{\pi}{12}$

(3) $\sin(3t) - \cos(3t) = \sqrt{2}\sin(3t - \frac{\pi}{4})$

周期 $\frac{2\pi}{3}$

振幅 $\sqrt{2}$

初期位相 $-\frac{\pi}{12}$

＜ 4 ページ. 異周期正弦波の和 ＞

問の解答

(1) 周期 2π

(2) 周期 2π

(3) 周期 $\dfrac{2\pi}{3}$

＜ 5ページ. 偶関数と奇関数 1 ＞

問の解答

(1) 偶関数

(2) 奇関数

(3) 偶関数

(4) 奇関数

(5) 偶関数

(6) 奇関数

(7) 偶関数

(8) 奇関数

(9) 偶関数

＜ 6 ページ. 偶関数と奇関数 2 ＞

問 1 の解答

(1) 偶関数

(2) 奇関数

(3) 偶関数

(4) 偶関数

(5) 偶関数

(6) 奇関数

(7) 偶関数

(8) 奇関数

(9) 偶関数

(10) 奇関数

(11) 偶関数

問 2 の解答

(1) 偶関数

(2) 偶関数

(3) 奇関数

(4) 奇関数

(5) 偶関数

< 7ページ. 三角多項式 1 >

問の解答

(図 4) の式は (3)

(図 5) の式は (4)

(図 5) の式は (5)

＜ 8 ページ. 三角多項式 2 ＞

問の解答

$f(t)$ の図は (図 6)

$g(t)$ の図は (図 5)

$h(t)$ の図は (図 4)

< 9ページ. 積分 1 >

問の解答

① $\displaystyle\int_{-\pi}^{\pi}\cos^2(nt)dt=\int_{-\pi}^{\pi}\left\{\frac{1}{2}+\frac{1}{2}\cos(2nt)\right\}dt=\left[\frac{t}{2}+\frac{1}{4n}\sin(2nt)\right]_{-\pi}^{\pi}=\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)=\pi$

② $\displaystyle\int_{-\pi}^{\pi}\sin^2(nt)dt=\int_{-\pi}^{\pi}\left\{\frac{1}{2}-\frac{1}{2}\cos(2nt)\right\}dt=\left[\frac{t}{2}-\frac{1}{4n}\sin(2nt)\right]_{-\pi}^{\pi}=\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)=\pi$

③ $\displaystyle\int_{-\pi}^{\pi}\frac{1}{2}\left\{\cos(n-m)t-\cos(n+m)t\right\}dt=\frac{1}{2}\left\{\frac{\sin(n-m)t}{n-m}-\frac{\sin(n+m)t}{n+m}\right\}_{-\pi}^{\pi}=0$

⑥ $\displaystyle\int_{-\pi}^{\pi}\frac{1}{2}\left\{\cos(n+m)t-\cos(n-m)t\right\}dt=\frac{1}{2}\left\{\frac{\sin(n+m)t}{n+m}+\frac{\sin(n-m)t}{n-m}\right\}_{-\pi}^{\pi}=0$

⑦ $\displaystyle\int_{-\pi}^{\pi}\cos(nt)dt=\left[\frac{1}{n}\sin(nt)\right]_{-\pi}^{\pi}=0$

＜ 10 ページ. 積分 2 ＞

問の解答

(1) 0

(2) 0

(3) $2\int_0^{\pi} \frac{1}{2}\left\{\cos t - \cos(7t)\right\} dt = \left[\sin t - \frac{1}{7}\sin(7t)\right]_0^{\pi} = 0$

＜ 11 ページ. 積分 3 ＞

問の解答

(1) $\left[+\dfrac{\sin(4t)}{4}\right]_0^{\pi} - \displaystyle\int_0^{\pi} \frac{1}{4}\sin(4t)dt = \left[\frac{1}{16}\cos(4t)\right]_0^{\pi} = \frac{1}{16}\{\cos(4\pi) - \cos 0\} = 0$

(2) $\left[t \times \left(-\dfrac{\cos(4t)}{4}\right)\right]_0^{\pi} + \displaystyle\int_0^{\pi} \frac{1}{4}\cos(4t)dt = -\frac{\pi}{4}\cos(4\pi) + \left[\frac{1}{16}\sin(4t)\right]_0^{\pi} = -\frac{\pi}{4}$

(3) $\left[t\dfrac{\sin(5t)}{5}\right]_0^{\pi} - \displaystyle\int_0^{\pi} \frac{1}{5}\sin(5t)dt = \left[\frac{1}{25}\cos(5t)\right]_0^{\pi} = -\frac{2}{25}$

(4) $\left[-t\dfrac{\cos(5t)}{5}\right]_0^{\pi} + \displaystyle\int_0^{\pi} \frac{1}{5}\cos(5t)dt = -\frac{\pi}{5}\cos(5\pi) + \left[\frac{1}{25}\sin(5t)\right]_0^{\pi} = -\frac{\pi}{5}$

< 12 ページ. 積分 4 >

問の解答

(1) $I_n = \displaystyle\int_0^{\pi} \sin(nt)dt = \left[-\frac{1}{n}\cos(nt)\right]_0^{\pi} = -\frac{1}{n}\{\cos(\pi n) - \cos 0\}$

① n が奇数のとき　$I_n = -\dfrac{1}{n}\{-1-1\} = \dfrac{2}{n}$

② n が偶数のとき　$I_n = -\dfrac{1}{n}\{1-1\} = 0$

(2) $I_n = \displaystyle\int_0^{\pi} t\sin(nt)dt = \left[-\frac{t}{n}\cos(nt)\right]_0^{\pi} + \int_0^{\pi}\frac{1}{n}\cos(nt)dt$

$$= -\frac{\pi}{n}\cos(n\pi) - 0 + \left[\frac{1}{n^2}\sin(n\pi)\right]_0^{\pi}$$

$$= -\frac{\pi}{n}\cos(n\pi)$$

① n が奇数のとき　$I_n = \dfrac{\pi}{n}$

② n が偶数のとき　$I_n = -\dfrac{\pi}{n}$

< 13ページ. 三角多項式の係数 1 >

問の解答

(1) πa_1

(2) πb_1

(3) πb_2

(4) πa_3

(5) πb_3

(6) 0

(7) 0

(8) 0

＜ 14ページ. 三角多項式の係数 2 ＞

問 1 の解答

(1) $a_1 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(t)\cos t dt$

(2) $b_1 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(t)\sin t dt$

(3) $b_2 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(t)\sin(2t) dt$

(4) $a_3 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(t)\cos(3t) dt$

(5) $b_3 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(t)\sin(3t) dt$

問 2 の解答

(1) $a_0 = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(t) dt$

(2) $a_k = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(t)\cos(kt) dt$

(3) $b_k = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(t)\sin(kt) dt$

< 15 ページ. フーリエ級数 1 >

問の解答

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt = 0$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)dt = 0$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)dt = \frac{2}{\pi}\int_{0}^{\pi} f(t)\sin(kt)dt$$

$$f(t) \sim \sum_{k=1}^{\infty} b_k \sin(kt)$$

< 16 ページ. フーリエ級数 2 >

問の解答

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt = \frac{1}{2\pi}\int_{-\pi}^{0} 1dt = \frac{1}{2}$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)dt = \frac{1}{\pi}\int_{-\pi}^{0}\cos(kt)dt = \frac{1}{\pi}\left[\frac{1}{k}\sin(kt)\right]_{-\pi}^{0} = 0$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)dt = \frac{1}{\pi}\int_{-\pi}^{0}\sin(kt)dt = \frac{1}{\pi}\left[-\frac{1}{k}\cos(kt)\right]_{-\pi}^{0}$$

$$= \frac{-1}{\pi k}\{\cos 0 - \cos(kt)\} = \begin{cases} -\frac{2}{\pi k}: & k\text{ が奇数のとき} \\ 0: & k\text{ が偶数のとき} \end{cases}$$

$$f(t) \sim \frac{1}{2} - \frac{2}{\pi}\sin t - \frac{2}{3\pi}\sin(3t) - \frac{2}{5\pi}\sin(5t) - \frac{2}{7\pi}\sin(7t) - \frac{2}{9\pi}\sin(9t) - \frac{2}{11\pi}\sin(11t) - \cdots$$

< 17 ページ. フーリエ級数 3 >

問の解答

$f(t)$ は奇関数より $a_0 = 0$, $a_k = 0$

$$b_k = \frac{2}{\pi}\int_0^{\pi} f(t)\sin(kt)dt = -\frac{2}{\pi}\int_0^{\pi} t\sin(kt)dt$$

$$= \begin{cases} -\frac{2}{\pi k} : & (k : \text{奇数}) \\ \frac{2}{\pi k} : & (k : \text{偶数}) \end{cases}$$

$$f(t) \sim -\frac{2}{1}\sin t + \frac{2}{2}\sin(2t) - \frac{2}{3}\sin(3t) + \frac{2}{4}\sin(4t) - \frac{2}{5}\sin(5t) + \frac{2}{6}\sin(6t) + \cdots$$

< 18ページ. フーリエ級数 4 >

問の解答

$f(t)$ は偶関数より $b_k = 0$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(t)dt = \frac{1}{\pi}\int_0^{\pi} tdt = \frac{1}{\pi}\left[\frac{t^2}{2}\right]_0^{\pi} = \frac{1}{\pi}\times\frac{\pi^2}{2} = \frac{\pi}{2}$$

$$a_k = \frac{2}{\pi}\int_0^{\pi} f(t)\cos(kt)dt = \frac{2}{\pi}\times\int_0^{\pi} t\cos(kt)dt$$

$$= \frac{2}{\pi}\times\begin{cases} -\frac{2}{k^2}: & (k:\text{奇数}) \\ 0: & (k:\text{偶数}) \end{cases}$$

$$f(t) \sim \frac{\pi}{2} - \frac{4}{\pi}\left\{\cos t + \frac{1}{9}\cos(3t) + \frac{1}{25}\cos(5t) + \frac{1}{49}\cos(7t) + \cdots\right\}$$

< 19ページ. フーリエ級数 5 >

問の解答

略

＜ 20 ページ. フーリエ級数 6 ＞

問の解答

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt = \frac{1}{2\pi}\int_{-1}^{1} 1dt = \frac{1}{\pi}$$

$$a_k = \frac{2}{\pi}\int_0^1 \cos(kt)dt = \frac{2}{\pi}\left[\frac{1}{k}\sin(kt)\right]_0^1 = \frac{2\sin k}{\pi k}$$

$$b_k = 0$$

$$S_n(t) = \frac{1}{\pi} + \frac{2}{\pi}\left\{\sum_{k=1}^{n}\frac{\sin k}{k}\cos(kt)\right\}$$

< 21 ページ. フーリエ級数 7 >

問の解答

$$f(0) = 1$$

$$S_\infty(0) = \frac{1}{2}$$

$$f(\pi) = 0$$

$$S_\infty(\pi) = \frac{1}{2}$$

＜ 22 ページ. 左極限・右極限 ＞

問 1 の解答

① $f_-(0) = -1$

② $f_+(0) = 0$

③ $f_-(2) = 1$

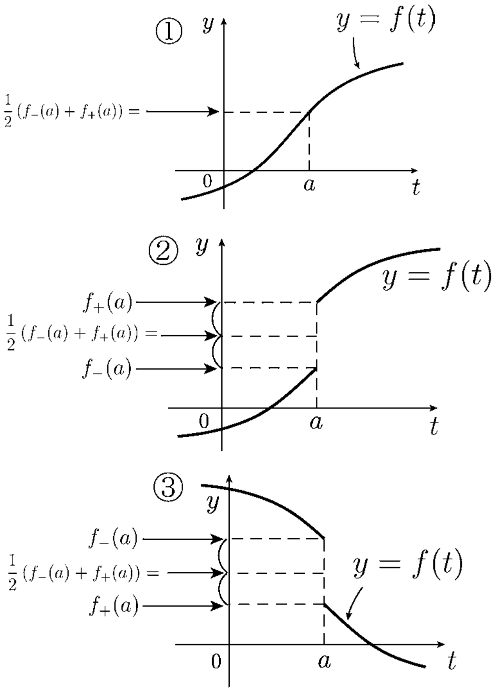
④ $f_+(2) = 2$

問 2 の解答

AB$= b - a$

$$m = a + \frac{1}{2}(b-a) = \frac{a+b}{2}$$

問 3 の解答



< 23ページ. フーリエ級数の収束 >

問の解答

(1) $S_\infty(\frac{3\pi}{2}) = \dfrac{\pi}{2}$

(2) $S_\infty(0) = 0$

(3) $S_\infty(\frac{\pi}{2}) = \dfrac{\pi}{4}$

< 24ページ. 一般の周期関数 1 >

問の解答

(1) 5

(2) 7

(3) 9

(4) 6

(5) 4

(6) 2

(7) $\dfrac{2}{3}$

(8) $\dfrac{2}{n}$

(9) L

(10) 2ℓ

(11) $\dfrac{L}{n}$

(12) $\dfrac{2\ell}{n}$

< 25 ページ. 一般の周期関数 2 >

問の解答

(1) 周期 5

(2) 周期 L

(3) 周期 2ℓ

< 26 ページ. 一般周期のフーリエ級数 1 >

問の解答

(1) $f(t) \sim a_0 + \displaystyle\sum_{k=1}^{\infty} \left\{ a_k \cos\left(\frac{k\pi}{\ell}t\right) + b_k \sin\left(\frac{k\pi}{\ell}t\right) \right\}$

$$\left(\begin{aligned} & a_0 = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(t)dt \ , \quad a_k = \frac{1}{\ell}\int_{-\ell}^{\ell} f(t)\cos\left(\frac{k\pi}{\ell}t\right)dt \\ & b_k = \frac{1}{\ell}\int_{-\ell}^{\ell} f(t)\sin\left(\frac{k\pi}{\ell}t\right)dt \quad (k \geqq 1) \end{aligned} \right)$$

(2) $f(t) \sim a_0 + \displaystyle\sum_{k=1}^{\infty} \left\{ a_k \cos\left(\frac{k}{\ell}t\right) + b_k \sin\left(\frac{k}{\ell}t\right) \right\}$

$$\left(\begin{aligned} & a_0 = \frac{1}{2\pi\ell}\int_{-\pi\ell}^{\pi\ell} f(t)dt \ , \quad a_k = \frac{1}{\pi\ell}\int_{-\pi\ell}^{\pi\ell} f(t)\cos\left(\frac{k}{\ell}t\right)dt \\ & b_k = \frac{1}{\pi\ell}\int_{-\pi\ell}^{\pi\ell} f(t)\sin\left(\frac{k}{\ell}t\right)dt \end{aligned} \right)$$

< 27 ページ. 一般周期のフーリエ級数 2 >

問 1 の解答

$$a_0 = 0$$

$$a_k = 0$$

$$b_k = \frac{4}{L}\int_0^{\frac{L}{2}} f(t)\sin\left(\frac{2k\pi}{L}t\right)dt$$

問 2 の解答

$$a_0 + \sum_{k=1}^{\infty}\left\{a_k\cos\left(\frac{k\pi}{\ell}t\right) + b_k\sin\left(\frac{k\pi}{\ell}t\right)\right\} = \frac{f_+(t)+f_-(t)}{2}$$

ただし

$$a_0 = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(t)dt\ ,\quad a_k = \frac{1}{\ell}\int_{-\ell}^{\ell} f(t)\cos\left(\frac{k\pi}{\ell}t\right)dt\ ,\quad b_k = \frac{1}{\ell}\int_{-\ell}^{\ell} f(t)\sin\left(\frac{k\pi}{\ell}t\right)dt$$

問 3 の解答

(1) $a_0 = \frac{1}{\ell}\int_0^{\ell} f(t)dt$

$$a_k = \frac{2}{\ell}\int_0^{\ell} f(t)\cos\left(\frac{k\pi}{\ell}t\right)dt$$

$$b_k = 0$$

(2) $a_0 = 0$

$$a_k = 0$$

$$a_k = \frac{2}{\ell}\int_0^{\ell} f(t)\sin\left(\frac{k\pi}{\ell}t\right)dt$$

< 28 ページ. オイラーの公式 >

問 1 の解答

(1) 1

(2) -1

(3) i

(4) $\dfrac{1+\sqrt{3}i}{2}$

(5) $\dfrac{\sqrt{3}-i}{2}$

(6) $-\dfrac{\sqrt{2}}{2}+\dfrac{\sqrt{2}}{2}i$

問 2 の解答

1

問 3 の解答

$e^{-i\theta}=\cos\theta-i\sin\theta$

問 4 の解答

$$\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$\sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}=\frac{i}{2}\left(e^{-i\theta}-e^{i\theta}\right)$$

< 29 ページ. 三角多項式の複素数表示 >

問の解答

$$S(t) = a_0 + \sum_{k=1}^{n} \left\{ a_k \frac{e^{ik\omega t} + e^{-ik\omega t}}{2} + b_k \times \frac{i}{2} \left(e^{-ik\omega t} - e^{ik\omega t} \right) \right\}$$

$$= a_0 + \sum_{k=1}^{n} \left(\frac{a_k}{2} - \frac{b_k}{2} i \right) e^{ik\omega t} + \sum_{k=1}^{n} \left(\frac{a_k}{2} + \frac{b_k}{2} i \right) e^{-ik\omega t}$$

ここで　$C_k = \frac{a_k}{2} - \frac{b_k}{2} i$，$C_{-k} = \frac{a_k}{2} + \frac{b_k}{2} i \ (k \geqq 1)$，$C_0 = a_0$ とおくと

$$S(t) = C_0 + \sum_{k=1}^{n} C_k \, e^{ik\omega t} + \sum_{k=1}^{n} C_{-k} \, e^{-ik\omega t} = \sum_{k=-n}^{n} C_k \, e^{ik\omega t}$$

＜ 30 ページ. フーリエ級数の複素数表示 1 ＞

問の解答

(1) $C_{-k} = \dfrac{1}{2}(a_k + b_k i) = \dfrac{1}{2}\left\{\left(\dfrac{2}{L}\displaystyle\int_{-\frac{L}{2}}^{\frac{L}{2}} f(t)\cos(k\omega t)dt\right) + \left(\dfrac{2}{L}\displaystyle\int_{-\frac{L}{2}}^{\frac{L}{2}} f(t)\sin(k\omega t)dt\right) i\right\}$

$$= \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(t)\left\{\cos(k\omega t) + i\sin(k\omega t)\right\}dt = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(t)e^{ik\omega t}dt$$

(2) $C_0 = a_0 = \dfrac{1}{L}\displaystyle\int_{-\frac{L}{2}}^{\frac{L}{2}} f(t)dt$

< 31 ページ. フーリエ級数の複素数表示 2 >

問 1 の解答

$$C_{-k} = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(t)e^{ik\omega t}dt = \frac{1}{2}(a_k + b_k i)$$

$$C_0 = a_0$$

問 2 の解答

(1) $f(t) \sim \displaystyle\sum_{k=-\infty}^{\infty} C_k\, e^{ikt}\ ,\quad C_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}dt$

(2) $f(t) \sim \displaystyle\sum_{k=-\infty}^{\infty} C_k\, e^{i\frac{k}{m}t}\ ,\quad C_k = \frac{1}{2\pi m}\int_{-\pi m}^{\pi m} f(t)e^{-i\frac{k}{m}t}dt$

< 32 ページ. 広義積分 1 >

問の解答

(1) $$\int_0^{\infty} e^{-\gamma t}dt = \lim_{b\to+\infty}\int_0^b e^{-\gamma t}dt = \lim_{b\to+\infty}\left[-\frac{1}{\gamma}e^{-\gamma t}\right]_0^b$$
$$= \lim_{b\to\infty}\left(-\frac{1}{\gamma}e^{-\gamma b} + \frac{1}{\gamma}\right) = \frac{1}{\gamma}$$

(2) $$\int_0^{\infty} \frac{1}{t^{\lambda}}dt = \lim_{b\to+\infty}\int_1^b t^{-\lambda}dt = \lim_{b\to+\infty}\left[\frac{1}{-\lambda+1}t^{-\lambda+1}\right]_1^b$$
$$= \lim_{b\to\infty} -\frac{1}{(\lambda-1)}\left[\frac{1}{b^{\lambda-1}} - 1\right] = \frac{1}{\lambda-1}$$

< 34 ページ. 広義積分の近似 >

問の解答

(1) $$\lim_{\Delta x \to 0} \sum_{k=0}^{\infty} \frac{\cos(\alpha k \Delta x)\Delta x}{1+(k\Delta x)^2} = \int_0^{\infty} \frac{\cos(\alpha x)}{1+x^2} dx$$

(2) $$\lim_{\Delta x \to 0} \sum_{k=-\infty}^{\infty} F(k\Delta x) e^{ik\Delta x t} \Delta x = \int_{-\infty}^{\infty} F(x) e^{ixt} dx$$

< 37ページ. フーリエ変換 1 >

問 1 の解答

$$\mathscr{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\left\{\cos(xt) - i\sin(xt)\right\} dt = -2i\int_{0}^{\infty} f(t)\sin(xt)dt$$

問 2 の解答

$$\mathscr{F}[f(t)] = 2\int_{0}^{\infty} f(t)\cos(xt)dt = 2\int_{0}^{T} 1\cos(xt)dt$$

$$= 2\left[\frac{1}{x}\sin(xt)\right]_{t=0}^{t=T} = \frac{2\sin(xT)}{x}$$

＜ 38 ページ. フーリエ変換 2 ＞

問の解答

$$\mathscr{F}\left[f(t)\right] = \int_{-\infty}^{\infty} f(t)e^{-ixt}dt = \int_{-\infty}^{0} e^{\alpha t}e^{-ixt}dt$$

$$= \lim_{a\to-\infty}\int_{a}^{0} e^{(\alpha-ix)t}dt = \lim_{a\to-\infty}\left[\frac{1}{\alpha-ix}e^{(\alpha-ix)t}\right]_{a}^{0}$$

$$= \lim_{a\to-\infty}\left\{\frac{1}{\alpha-ix} - \frac{e^{(\alpha-ix)a}}{\alpha-ix}\right\} = \frac{1}{\alpha-ix}$$

＜ 39 ページ. フーリエ変換 3 ＞

問の解答

$$\mathscr{F}\left[e^{-\alpha|t|}\right] = \int_{-\infty}^{\infty} e^{-\alpha|t|}e^{-ixt}dt = \lim_{\substack{b\to+\infty \\ a\to-\infty}} \int_a^b e^{-\alpha|t|-ixt}dt$$

$$= \lim_{\substack{b\to+\infty \\ a\to-\infty}} \left\{\int_a^0 e^{(\alpha-ix)t}dt + \int_0^b e^{-(\alpha+ix)t}dt\right\}$$

$$= \lim_{\substack{b\to+\infty \\ a\to-\infty}} \left\{\left[\frac{1}{\alpha-ix}e^{(\alpha-ix)t}\right]_a^0 + \left[\frac{1}{-(\alpha+ix)}e^{-(\alpha-ix)t}\right]_0^b\right\}$$

$$= \lim_{\substack{b\to+\infty \\ a\to-\infty}} \left\{\frac{1}{\alpha-ix}\left(1-e^{(\alpha-ix)a}\right) - \frac{1}{\alpha+ix}\left(e^{-(\alpha+ix)b}-1\right)\right\}$$

$$= \frac{1}{\alpha-ix} + \frac{1}{\alpha+ix} = \frac{\alpha+ix+\alpha-ix}{(\alpha-ix)(\alpha+ix)} = \frac{2\alpha}{\alpha^2+x^2}$$

＜ 42 ページ. フーリエ変換 6 ＞

問の解答

$$\mathscr{F}\left[e^{-\alpha t^2}\right] = \int_{-\infty}^{\infty} e^{-\alpha t^2} e^{-ixt} dt = F(x) \text{ とおく}$$

$$\frac{d}{dx}F(x) = \int_{-\infty}^{\infty} e^{-\alpha t^2} \frac{d}{dx}\left(e^{-ixt}\right) dt = \int_{-\infty}^{\infty} e^{-\alpha t^2}(-it)e^{-ixt}dt$$

$$= \frac{i}{2\alpha}\int_{-\infty}^{\infty}\left(-2\alpha t e^{-\alpha t^2}\right) e^{-ixt}dt = \frac{i}{2\alpha}\int_{-\infty}^{\infty}\left(e^{-\alpha t^2}\right)' e^{-ixt}dt$$

$$= \frac{i}{2\alpha}\mathscr{F}\left[\left(e^{-\alpha t^2}\right)'\right] = \frac{i}{2\alpha} \times ix\mathscr{F}\left[e^{-\alpha t^2}\right] = -\frac{x}{2\alpha}\mathscr{F}\left[e^{-\alpha t^2}\right]$$

$$= -\frac{x}{2\alpha}F(x)$$

$$\Downarrow$$

$$\frac{d}{dx}F(x) = -\frac{x}{2\alpha}F(x) \quad \Rightarrow \quad F(x) = Ce^{-\frac{x^2}{4\alpha}}$$

$$F(0) = \int_{-\infty}^{\infty} e^{-\alpha t^2}dt = \int_{-\infty}^{\infty} e^{-u^2}\frac{du}{\sqrt{\alpha}} = \frac{1}{\sqrt{\alpha}}\int_{-\infty}^{\infty} e^{-u^2}du \qquad \left(u = \sqrt{\alpha}\,t \quad dt = \frac{du}{\sqrt{\alpha}}\right)$$

$$= \frac{\sqrt{\pi}}{\sqrt{\alpha}} = C$$

$$\Rightarrow \quad F(x) = \sqrt{\frac{\pi}{\alpha}}\,e^{-\frac{x^2}{4\alpha}}$$

< 47 ページ. デルタ関数 >

問 1 の解答

$(f * \delta)(t) = \delta(t)$

問 2 の解答

$f(2) = 2^3 = 8$

問 3 の解答

$\sin\left(\frac{\pi}{2}\right) = 1$

問 4 の解答

$(f * \delta)(3) = 2.5$

< 49 ページ. フーリエ逆変換 2 >

問 1 の解答

(1) $\mathscr{F}^{-1}\left[\dfrac{4}{x^2+4}\right]=e^{-2|t|}$

(2) $\mathscr{F}^{-1}\left[\sqrt{\pi}e^{-\frac{x^2}{4}}\right]=e^{-t^2}$

(3) $\mathscr{F}^{-1}\left[\dfrac{2\sin(4x)}{x}\right]=\begin{cases}1 & : \quad |t|\leqq 4\\ 0 & : \quad |t|>4\end{cases}$

問 2 の解答

$$\begin{aligned}\mathscr{F}^{-1}\left[\frac{3}{x^2+1}+e^{-x^2}\right]&=\frac{3}{2}\mathscr{F}^{-1}\left[\frac{2}{x^2+1}\right]+\frac{1}{\sqrt{4\pi}}\mathscr{F}^{-1}\left[\sqrt{4\pi}e^{-x^2}\right]\\&=\frac{3}{2}e^{-|t|}+\frac{1}{\sqrt{4\pi}}e^{-\frac{t^2}{4}}\end{aligned}$$

< 50ページ. 超関数のフーリエ関数 >

問の解答

$$\mathscr{F}[\cos(\alpha t)] = \frac{1}{2}\left\{\mathscr{F}\left[e^{i\alpha t}\right] + \mathscr{F}\left[e^{-i\alpha t}\right]\right\}$$

$$= \pi\delta(x-\alpha) + \pi\delta(x+\alpha)$$

$$\mathscr{F}\left[\sin(\alpha t)\right] = \frac{1}{2i}\left\{\mathscr{F}\left[e^{i\alpha t}\right] - \mathscr{F}\left[e^{-i\alpha t}\right]\right\}$$

$$= -i\pi\delta(x-\alpha) + i\pi\delta(x+\alpha)$$

< 51 ページ. 周波数関数 >

問の解答

時間関数 $f(t)$	周波数関数 $F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$
$a_1f_1(t) + a_2f_2(t)$	$a_1F_1(\omega) + a_2F_2(\omega)$
$F(t)$	$2\pi f(-\omega)$
$f(at)$ $(a \neq 0)$	$\frac{1}{a}F\left(\frac{\omega}{a}\right)$
$f(t-t_0)$	$e^{-it_0\omega}F(\omega)$
$f(t)e^{i\omega_0 t}$	$F(\omega-\omega_0)$
$f(t)e^{-i\omega_0 t}$	$F(\omega+\omega_0)$
$\frac{d^n}{dt^n}f(t)$ (n 回微分)	$(i\omega)^nF(\omega)$
$\int_{-\infty}^{t} f(\tau)d\tau$	$\frac{1}{i\omega}F(\omega)$
$(-it)^nf(t)$	$\frac{d^n}{d\omega^n}F(\omega)$ (n 回微分)
$(f_1 * f_2)(t)$ (合成積)	$F_1(\omega)\times F_2(\omega)$
$f_1(t)f_2(t)$ (積)	$\frac{1}{2\pi}(F_1 * F_2)(\omega)$
$f(t) = \begin{cases} 1 & : \ \lvert t\rvert \leqq T \\ 0 & : \ \lvert t\rvert > T \end{cases}$	$\frac{2\sin(\omega T)}{\omega}$ $(T > 0)$
$f(t) = \begin{cases} e^{-\alpha t} & : \ t > 0 \\ 0 & : \ t \leqq 0 \end{cases}$ $(\alpha > 0)$	$\frac{1}{\alpha + i\omega}$
$e^{-\alpha\lvert t\rvert}$	$\frac{2\alpha}{\omega^2+\alpha^2}$ $(\alpha > 0)$
$e^{-\alpha t^2}$ $(\alpha > 0)$	$\sqrt{\frac{\pi}{\alpha}}e^{-\frac{\omega^2}{4\alpha}}$
$\frac{1}{\sqrt{4b\pi}}e^{-\frac{t^2}{4b}}$	$e^{-b\omega^2}$ $(b > 0)$
$\frac{b}{\pi(t^2+b^2)}$	$e^{-b\lvert\omega\rvert}$ $(b > 0)$
$\delta(t)$ (デルタ関数)	1
$\delta(t-t_0)$	$e^{-i\omega t_0}$
$e^{i\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$

＜53 ページ. ラプラス変換 1 ＞

問の解答

(1) $$\mathscr{L}[1] = \int_0^\infty 1e^{-st}dt = \left[\frac{1}{-s}e^{-st}\right]_0^\infty = \lim_{n\to\infty}\left\{-\frac{1}{se^{ns}} + \frac{1}{s}\right\}$$
$$= \frac{1}{s}$$

(2) $$\mathscr{L}[e^{-t}] = \int_0^\infty e^{-t}e^{-st}dt = \left[-\frac{1}{s+1}e^{-(s+1)t}\right]_0^\infty = \frac{1}{s+1}$$

＜ 54 ページ. ラプラス変換 2 ＞

問の解答

(1) $\mathscr{L}[\sin(kt)] = \mathscr{L}\left[\dfrac{e^{ikt}-e^{-ikt}}{2i}\right] = \dfrac{1}{2i}\left\{\mathscr{L}\left[e^{ikt}\right] - \mathscr{L}\left[e^{-ikt}\right]\right\}$

$$= \frac{1}{2i}\left\{\frac{1}{s-ik} - \frac{1}{s+ik}\right\} = \frac{1}{2i} \times \frac{s+ik-(s-ik)}{(s-ik)(s+ik)}$$

$$= \frac{1}{2i} \times \frac{2ik}{s^2+k^2} = \frac{k}{s^2+k^2}$$

(2) $\mathscr{L}\left[e^{\alpha t}\cos(kt)\right] = \mathscr{L}\left[e^{\alpha t}\dfrac{e^{ikt}+e^{-ikt}}{2}\right] = \dfrac{1}{2}\mathscr{L}\left[e^{\alpha t}e^{ikt}\right] + \dfrac{1}{2}\mathscr{L}\left[e^{\alpha t}e^{-ikt}\right]$

$$= \frac{1}{2} \times \frac{1}{(s-\alpha)-ik} + \frac{1}{2} \times \frac{1}{(s-\alpha)+ikt}$$

$$= \frac{2(s-\alpha)}{2\left\{(s-\alpha)^2+k^2\right\}} = \frac{s-\alpha}{(s-\alpha)^2+k^2}$$

(3) $\mathscr{L}\left[e^{\alpha t}\sin kt\right] = \dfrac{1}{2i}\mathscr{L}\left[e^{(\alpha+ik)t} - e^{(\alpha-ik)t}\right] = \dfrac{1}{2i} \times \left\{\dfrac{1}{(s-\alpha)-ik} - \dfrac{1}{(s-\alpha)+ik}\right\}$

$$= \frac{k}{(s-\alpha)^2+k^2}$$

＜ 55 ページ. ラプラス変換 3 ＞

問の解答

(1) $\mathscr{L}\left[t^3\right] = -\dfrac{d^3}{ds^3}\ \dfrac{1}{s} = -\dfrac{d}{ds}\left(\dfrac{2}{s^3}\right) = \dfrac{6}{s^4}$

$$\left(\begin{array}{c} F(s) = \mathscr{L}(1) = \dfrac{1}{s}\ \ ,\ \ \mathscr{L}(t) = -\dfrac{d}{ds}F(s) = -\left(\dfrac{1}{s}\right)' = \dfrac{1}{s^2}\ , \\ \mathscr{L}\left[t^2\right] = \dfrac{d^2}{ds^2}\left(\dfrac{1}{s}\right) = \left(-\dfrac{1}{s^2}\right)' = \dfrac{2}{s^3} \end{array}\right)$$

(2) $\mathscr{L}\left[t^4\right] = \dfrac{d^4}{ds^4}\left(\dfrac{1}{s}\right) = \left(-\dfrac{6}{s^4}\right)' = \dfrac{6\times 4}{s^5} = \dfrac{24}{s^5}$

(3) $\mathscr{L}\left[t^n\right] = \dfrac{n!}{s^{n+1}}$

(4) $\mathscr{L}\left[e^{at}\right] = \displaystyle\int_0^\infty e^{at}e^{-st}dt = \int_0^\infty e^{(a-s)t}dt = \left[\frac{1}{a-s}e^{(a-s)t}\right]_0^\infty = \frac{1}{s-a}$

(5) $\mathscr{L}\left[te^{at}\right] = -\left(\dfrac{1}{s-a}\right)' = -\dfrac{-1}{(s-a)^2} = \dfrac{1}{(s-a)^2}$

(6) $\mathscr{L}[t\cos(kt)] = -\left(\dfrac{s}{s^2+k^2}\right)' = -\dfrac{1\times(s^2+k^2) - s\times 2s}{(s^2+k^2)^2} = \dfrac{s^2-k^2}{(s^2+k^2)^2}$

(7) $\mathscr{L}[t\sin(kt)] = -\left(\dfrac{k}{s^2+k^2}\right)' = -\dfrac{0-k\times 2s}{(s^2+k^2)^2} = \dfrac{2ks}{(s^2+k^2)^2}$

(8) $\mathscr{L}[\sinh(kt)] = \mathscr{L}\left[\dfrac{1}{2}\left(e^{kt}-e^{-kt}\right)\right] = \dfrac{1}{2}\mathscr{L}\left[e^{kt}\right] - \dfrac{1}{2}\mathscr{L}\left[e^{-kt}\right]$

$$= \frac{1}{2}\times\left\{\frac{1}{s-k} - \frac{1}{s+k}\right\} = \frac{k}{s^2-k^2}$$

(9) $\mathscr{L}[\cosh(kt)] = \mathscr{L}\left[\dfrac{1}{2}\left(e^{kt}+e^{-kt}\right)\right]$

$$= \frac{1}{2}\left\{\mathscr{L}\left[e^{kt}\right] + \mathscr{L}\left[e^{-kt}\right]\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-k} + \frac{1}{s+k}\right\} = \frac{s}{s^2-k^2}$$

< 56 ページ. ラプラス変換 4 >

問の解答

$$\mathscr{L}\left[f'''(t)\right] = \mathscr{L}\left[(f''(t))'\right] = s\left\{\mathscr{L}\left[f''(t)\right]\right\} - f''(+0)$$

$$= s\left\{s^2F(s) - sf(+0) - f'(+0)\right\} - f''(+0)$$

$$= s^3F(s) - s^2f(+0) - sf'(+0) - f''(+0)$$

$<$ 60ページ.ラプラス変換 8 $>$

問の解答

原関数 $f(t)$	像関数 $\mathscr{L}[f(t)] = F(s)$
1	$\dfrac{1}{s}$
t	$\dfrac{1}{s^2}$
t^n	$\dfrac{n!}{s^{n+1}}$
e^{at}	$\dfrac{1}{s-a}$
te^{at}	$\dfrac{1}{(s-a)^2}$
t^2e^{at}	$-\left(\dfrac{1}{(s-a)^2}\right)' = -\dfrac{-2}{(s-a)^3} = \dfrac{2}{(s-a)^3}$
$\sin(\omega t)$	$\dfrac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\dfrac{s}{s^2+\omega^2}$
$e^{at}\sin(\omega t)$	$\dfrac{\omega}{(s-\alpha)^2+\omega^2}$
$e^{at}\cos(\omega t)$	$\dfrac{s-\alpha}{(s-\alpha)^2+\omega^2}$
$t\sin(\omega t)$	$\dfrac{2\omega s}{(s^2+\omega^2)^2}$
$t\cos(\omega t)$	$\dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$
$\sinh(kt) = \dfrac{1}{2}\left(e^{kt}-e^{-kt}\right)$	$\dfrac{k}{s^2-k^2}$
$\cosh(kt) = \dfrac{1}{2}\left(e^{kt}+e^{-kt}\right)$	$\dfrac{s}{s^2-k^2}$
$u(t-a) = \begin{cases} 1 & : \ t > a \\ 0 & : \ t \leqq a \end{cases}$ $(a > 0)$	$\mathscr{L}[u(t-a)] = \displaystyle\int_a^\infty 1e^{-st}dt$ $= \left[-\dfrac{1}{s}e^{-st}\right]_a^\infty = \dfrac{1}{s}e^{-sa}$
$\dfrac{a}{2\pi t^{\frac{3}{2}}}e^{-\frac{a^2}{4t}}$	$\sqrt{\pi}e^{-a\sqrt{s}}$

< 62 ページ. ラプラス逆変換 2 >

問の解答

$F(s)$	$\mathscr{L}^{-1}[F(s)] = f(t)$
$\dfrac{1}{s}$	1
$\dfrac{1}{s^2}$	t
$\dfrac{n!}{s^{n+1}}$	t^n
$\dfrac{1}{s-a} \qquad (s > \lvert a \rvert)$	e^{at}
$\dfrac{1}{(s-a)^2}$	te^{at}
$\dfrac{\omega}{s^2+\omega^2}$	$\sin(\omega t)$
$\dfrac{s}{s^2+\omega^2}$	$\cos(\omega t)$
$\dfrac{\omega}{(s-a)^2+\omega^2}$	$e^{at}\sin(\omega t)$
$\dfrac{s-a}{(s-a)^2+\omega^2}$	$e^{at}\cos(\omega t)$
$\dfrac{2\omega s}{(s^2+\omega^2)^2}$	$t\sin(\omega t)$
$\dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$	$t\cos(\omega t)$
$\dfrac{\omega}{s^2-\omega^2}$	$\sinh(\omega t)$
$\dfrac{s}{s^2-\omega^2}$	$\cosh(\omega t)$
$\dfrac{e^{-\alpha s}}{s}$	$f(t) = \begin{cases} 1 & : \ t > a \\ 0 & : \ t \leqq a \end{cases}$
$e^{-\alpha\sqrt{s}}$	$\dfrac{\alpha}{2\sqrt{\pi}t^{\frac{3}{2}}}e^{-\frac{\alpha^2}{4t}}$

< 63 ページ. ラプラス逆変換 3 >

問の解答

(1) $$\mathscr{L}^{-1}\left[\frac{1}{s^2-s-2}\right]=\frac{1}{3}\left\{\mathscr{L}^{-1}\left[\frac{1}{s-2}\right]-\mathscr{L}^{-1}\left[\frac{1}{s+1}\right]\right\}$$
$$=\frac{1}{3}\left\{e^{2t}-e^{-t}\right\}$$

(2) $$\mathscr{L}^{-1}\left[\frac{1}{s^2-4}\right]=\frac{1}{4}\left\{\mathscr{L}^{-1}\left[\frac{1}{s-2}\right]-\mathscr{L}^{-1}\left[\frac{1}{s+2}\right]\right\}$$
$$=\frac{1}{4}\left\{e^{2t}-e^{-2t}\right\}$$

(3) $$\mathscr{L}^{-1}\left[\frac{1}{s^2-4s+5}\right]=\mathscr{L}^{-1}\left[\frac{1}{(s-2)^2+1}\right]$$
$$=e^{2t}\sin(t)$$

＜ 64 ページ. ラプラス逆変換 4 ＞

問の解答

(1) $\mathscr{L}^{-1}\left[\dfrac{s-3}{s^2-8s+16}\right]=\mathscr{L}^{-1}\left[\dfrac{s-4+1}{(s-4)^2}\right]$

$=\mathscr{L}^{-1}\left[\dfrac{1}{s-4}+\dfrac{1}{(s-4)^2}\right]=e^{4t}+te^{4t}$

(2) $\mathscr{L}^{-1}\left[\dfrac{s+1}{s^2-6s+9}\right]=\mathscr{L}^{-1}\left[\dfrac{s-3+4}{(s-3)^2}\right]$

$=\mathscr{L}^{-1}\left[\dfrac{1}{s-3}+4\times\dfrac{1}{(s-3)^2}\right]=e^{3t}+4te^{3t}$

(3) $\mathscr{L}^{-1}\left[\dfrac{s+2}{s^2-2s+5}\right]=\mathscr{L}^{-1}\left[\dfrac{s-1+3}{(s-1)^2+4}\right]$

$=\mathscr{L}^{-1}\left[\dfrac{s-1}{(s-1)^2+2^2}+\dfrac{3}{2}\times\dfrac{2}{(s-1)^2+2^2}\right]$

$=e^t\cos(2t)+\dfrac{3}{2}e^t\sin 2t$

(4) $\mathscr{L}^{-1}\left[\dfrac{2s}{s^2-4s+5}\right]=\mathscr{L}^{-1}\left[\dfrac{2(s-2)+4}{(s-2)^2+1}\right]$

$=\mathscr{L}^{-1}\left[2\times\dfrac{s-2}{(s-2)^2+1}+4\times\dfrac{1}{(s-2)^2+1}\right]$

$=2e^{2t}\cos t+4e^{2t}\sin t$

< 65 ページ. ラプラス逆変換 5 >

問 1 の解答

(1) $\mathscr{L}^{-1}\left[\dfrac{1}{s(s-1)(s-2)}\right]=\dfrac{1}{2}\mathscr{L}^{-1}\left[\dfrac{1}{s}-\dfrac{2}{s-1}+\dfrac{1}{s-2}\right]$

$=\dfrac{1}{2}\left\{1-2e^{t}+e^{2t}\right\}$

(2) $\mathscr{L}^{-1}\left[\dfrac{s^2+s}{(s-1)(s+3)^2}\right]=\mathscr{L}^{-1}\left[\dfrac{1}{8}\times\dfrac{1}{s-1}+\dfrac{7}{8}\times\dfrac{1}{s+3}-\dfrac{3}{2}\times\dfrac{1}{(s+3)^2}\right]$

$=\dfrac{1}{8}e^{t}+\dfrac{7}{8}e^{-3t}-\dfrac{3}{2}te^{-3t}$

問 2 の解答

(1) $\mathscr{L}^{-1}\left[\dfrac{F(s)}{(s-a)^2}\right]=(te^{at}*f)(t)=\displaystyle\int_0^t(t-u)e^{a(t-u)}f(u)du$

(2) $\mathscr{L}^{-1}\left[\dfrac{F(s)}{(s-a)(s-b)}\right]=\mathscr{L}^{-1}\left[F(s)\times\dfrac{1}{a-b}\left\{\dfrac{1}{s-a}-\dfrac{1}{s-b}\right\}\right]$

$=\left(\dfrac{1}{a-b}\left\{e^{at}-e^{bt}\right\}*f\right)(t)=\displaystyle\int_0^t\dfrac{1}{a-b}\left\{e^{a(t-u)}-e^{b(t-u)}\right\}f(u)du$

(3) $\mathscr{L}^{-1}\left[\dfrac{F(s)}{(s-a)^2+b^2}\right]=\mathscr{L}^{-1}\left[F(s)\times\dfrac{1}{b}\times\dfrac{b}{(s-a)^2+b^2}\right]$

$=\left(\dfrac{1}{b}e^{at}\sin(bt)*f\right)(t)=\displaystyle\int_0^t\dfrac{1}{b}e^{a(t-u)}\sin(b(t-u))f(u)du$

＜ 66 ページ. ラプラス逆変換 6 ＞

問の解答

(1) $\mathscr{L}^{-1}\left[\frac{a}{s}+\frac{b}{s^2}+\frac{c}{s^3}\right]=a+bt+\frac{c}{2}t^2$

(2) $\mathscr{L}^{-1}\left[\frac{a+bs}{s^2+1}\right]=a\times\mathscr{L}^{-1}\left[\frac{1}{s^2+1}\right]+b\times\mathscr{L}^{-1}\left[\frac{s}{s^2+1}\right]=a\sin(t)+b\cos(t)$

(3) $\mathscr{L}^{-1}\left[\frac{s}{s^2+4}\right]=\cos(2t)$

(4) $\mathscr{L}^{-1}\left[\frac{1}{s^2-1}\right]=\frac{1}{2}\mathscr{L}^{-1}\left[\frac{1}{s-1}-\frac{1}{s+1}\right]=\frac{1}{2}\{e^t-e^{-t}\}\qquad(=\sinh(t))$

(5) $\mathscr{L}^{-1}\left[\frac{s-1}{(s-2)^2}\right]=\mathscr{L}^{-1}\left[\frac{s-2+1}{(s-2)^2}\right]=\mathscr{L}^{-1}\left[\frac{1}{s-2}+\frac{1}{(s-2)^2}\right]$

$=e^{2t}+te^{2t}$

(6) $\mathscr{L}^{-1}\left[\frac{s+3}{(s+1)(s-2)}\right]=\frac{1}{3}\mathscr{L}^{-1}\left[\frac{5}{s-2}-\frac{2}{s+1}\right]$

$=\frac{5}{3}e^{2t}-\frac{2}{3}e^{-t}$

(7) $\mathscr{L}^{-1}\left[\frac{-2}{(s-2)^3}\right]=\mathscr{L}^{-1}\left[\frac{d}{ds}\left(\frac{1}{(s-2)^2}\right)\right]=t^2e^{2t}$

(8) $\mathscr{L}^{-1}\left[\frac{16}{s^4-16}\right]=\frac{1}{2}\mathscr{L}^{-1}\left[\frac{1}{s-2}-\frac{1}{s+2}-\frac{4}{s^2+4}\right]$

$=\frac{1}{2}\left\{e^{2t}-e^{-2t}-2\sin(2t)\right\}$

＜67ページ. ラプラス逆変換 7＞

問の解答

(1) $f(t) = \dfrac{1}{2}\sin(2t)$, $g(t) = \cos(2t)$

(2) $(f*g)(t) = \displaystyle\int_0^t \frac{1}{2}\sin(2t-2u)\cos(2u)du = \frac{1}{4}\int_0^t \{\sin(2t)+\sin(2t-4u)\}du$

$$= \frac{1}{4}\left[u\sin(2t)+\frac{1}{4}\cos(4u-2t)\right]_0^t = \frac{1}{4}\left\{t\sin(2t)+\frac{1}{4}\cos(2t)-\frac{1}{4}\cos(-2t)\right\}$$

$$= \frac{1}{4}t\sin(2t)$$

(3) $\mathscr{L}^{-1}\left[\dfrac{1}{s^2+4}\times\dfrac{s}{s^2+4}\right] = (f*g)(t) = \dfrac{1}{4}t\sin 2t$

(4) $(g*g)(t) = \displaystyle\int_0^t \cos(2t-2u)\cos(2u)du = \frac{1}{2}\int_0^t \{\cos(2t)+\cos(2t-4u)\}du$

$$= \frac{1}{2}\int_0^t \{\cos(2t)+\cos(4u-2t)\}du = \frac{1}{2}\left[u\cos(2t)+\frac{1}{4}\sin(4u-2t)\right]_0^t$$

$$= \frac{1}{2}\left\{t\cos(2t)+\frac{1}{4}\sin(2t)-\frac{1}{4}\sin(-2t)\right\} = \frac{1}{2}t\cos(2t)+\frac{1}{4}\sin(2t)$$

(5) $\mathscr{L}^{-1}\left[\dfrac{s^2}{(s^2+4)^2}\right] = (g*g)(t)$

$$= \frac{1}{2}t\cos(2t)+\frac{1}{4}\sin(2t)$$

＜ 68 ページ. 常微分方程式への応用 1 ＞

問の解答

(1) $\mathscr{L}[x(t)] = X(s)$ とおくと $\mathscr{L}\left[\dfrac{dx}{dt}\right] = sX(s) - x(0) = sX(s) - a$

$$\mathscr{L}\left[\frac{dx}{dt}\right] = \mathscr{L}[kx] \ \Rightarrow\ sX(s) - a = kX(s) \ \Rightarrow\ (s-k)X(s) = a$$

$$X(s) = \frac{a}{s-k} \ \Rightarrow\ x(t) = \mathscr{L}^{-1}[X(s)] = \mathscr{L}^{-1}\left[\frac{a}{s-k}\right] = ae^{kt}$$

(2) 解 $x(t)$ に対し $\mathscr{L}[x(t)] = X(s)$ とおくと

$$\mathscr{L}\left[\frac{dx}{dt}\right] = sX(s) - x(0) = sX(s) - 1$$

$$\mathscr{L}\left[\frac{dx}{dt} + x\right] = \mathscr{L}\left[e^{-t}\right] \ \Rightarrow\ sX(s) - 1 + X(s) = \frac{1}{s+1}$$

$$\Rightarrow\ (s+1)X(s) = \frac{1}{s+1} + 1$$

$$X(s) = \frac{1}{(s+1)^2} + \frac{1}{s+1}$$

$$x(t) = \mathscr{L}^{-1}[X(s)] = \mathscr{L}^{-1}\left[\frac{1}{(s+1)^2} + \frac{1}{s+1}\right]$$

$$= te^{-t} + e^{-t}$$

< 69 ページ. 常微分方程式への応用 2 >

問の解答

(1) 解 $x(t)$ に対し $\mathscr{L}[x(t)] = X(s)$ とおくと

$$\mathscr{L}\left[\frac{dx}{dt}\right] = sX(s) - x(0) = sX(s)\ ,\ \mathscr{L}\left[\frac{d^2x}{dt^2}\right] = s^2X(s) - sx(0) - x'(0) = s^2X(s)$$

両辺をラプラス変換すると $s^2X(s) - 5sX(s) + 6X(s) = \dfrac{1}{s-1}$

$$X(s) = \frac{1}{(s^2-5s+6)(s-1)} = \frac{1}{(s-3)(s-2)(s-1)} = \frac{1}{2}\left\{\frac{1}{s-1} - \frac{2}{s-2} + \frac{1}{s-3}\right\}$$

$$x(t) = \mathscr{L}^{-1}[X(s)] = \frac{1}{2}e^t - e^{2t} + \frac{1}{2}e^{3t}$$

(2) 解 $x(t)$ のラプラス変換を $\mathscr{L}[x(t)] = X(s)$ とおく

$$\mathscr{L}\left[\frac{dx}{dt}\right] = sX(s) - x(0) = sX(s) - 1\ ,$$

$$\mathscr{L}\left[\frac{d^2x}{dt^2}\right] = s^2X(s) - sx(0) - x'(0) = s^2X(s) - s - 1$$

よって (2) のラプラス変換は

$$s^2X(s) - s - 1 - 4(sX(s) - 1) + 4X(s) = 0$$

$$(s^2 - 4s + 4)X(s) = s - 3$$

$$X(s) = \frac{s-3}{s^2-4s+4} = \frac{s-2-1}{(s-2)^2} = \frac{1}{s-2} - \frac{1}{(s-2)^2}$$

$$x(t) = \mathscr{L}^{-1}[X(s)] = e^{2t} - te^{2t}$$

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問の解答

(1) 解 $x(t)$ に対し $\mathscr{L}[x(t)] = X(s)$ とおく。$\mathscr{L}\left[\dfrac{dx}{dt}\right] = sX(s) - x(0) = sX(s)$

$$\mathscr{L}\left[\frac{d^2x}{dt^2}\right] = s^2X(s) - sx(0) - x'(0) = s^2X(s) - 1$$

(1) のラプラス変換は

$$s^2X(s) - 1 - 2sX(s) + 5X(s) = 0$$

$$(s^2 - 2s + 5)X(s) = 1 \ \Rightarrow\ X(s) = \frac{1}{s^2 - 2s + 5} = \frac{1}{(s-1)^2 + 2^2}$$

$$x(t) = \mathscr{L}^{-1}[X(s)] = \frac{1}{2}\mathscr{L}^{-1}\left[\frac{2}{(s-1)^2 + 2^2}\right] = \frac{1}{2}e^t\sin(2t)$$

(2) 解 $x(t)$ に対し $\mathscr{L}[x(t)] = X(s)$ とおく

$$\frac{d^2x}{dt^2} = s^2X(s) - sx(0) - x'(0) = s^2X(s) - s - 1$$

$$\mathscr{L}[2\sin t] = \frac{2}{s^2+1}$$

より (2) のラプラス変換は

$$s^2X(s) - s - 1 + 4X(s) = \frac{2}{s^2+1}$$

$$(s^2+4)X(s) = s + 1 + \frac{2}{s^2+1}$$

$$X(s) = \frac{s+1}{s^2+4} + \frac{2}{(s^2+1)(s^2+4)} = \frac{s}{s^2+4} + \frac{1}{s^2+4} + \frac{\frac{2}{3}}{s^2+1} - \frac{\frac{2}{3}}{s^2+4}$$

$$= \frac{s}{s^2+4} + \frac{\frac{1}{3}}{s^2+4} + \frac{\frac{2}{3}}{s^2+1}$$

$$x(t) = \mathscr{L}^{-1}[X(s)] = \mathscr{L}^{-1}\left[\frac{s}{s^2+4}\right] + \frac{1}{6}\mathscr{L}^{-1}\left[\frac{2}{s^2+4}\right] + \frac{2}{3}\mathscr{L}^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= \cos(2t) + \frac{1}{6}\sin(2t) + \frac{2}{3}\sin t$$

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問の解答

(1) $\mathscr{L}[x(t)] = X(s)$, $\mathscr{L}[f(t)] = F(s)$ とおき，両辺をラプラス変換する

$$s^2X(s) - 3sX(s) + 2X(s) = F(s)$$

$$X(s) = \frac{F(s)}{s^2-3s+2} \qquad \left(\frac{1}{s^2-3s+2} = \frac{1}{(s-1)(s-2)} = \frac{1}{s-2} - \frac{1}{s-1}\right)$$

$\mathscr{L}^{-1}\left[\dfrac{1}{s^2-3s+2}\right] = e^{2t} - e^t$ より

$$x(t) = \mathscr{L}^{-1}\left[\frac{1}{s^2-3s+2} \times F(s)\right] = (e^{2t}-e^t) * f(t) = \int_0^t \left\{e^{2(t-u)} - e^{t-u}\right\} f(u)du$$

(2) $\mathscr{L}[x(t)] = X(s)$, $\mathscr{L}[f(t)] = F(s)$ とおき，両辺をラプラス変換すると

$$s^2X(s) + 6sX(s) + 10X(s) = F(s)$$

$$X(s) = \frac{F(s)}{s^2+6s+10}$$

$$\frac{1}{s^2+6s+10} = \frac{1}{(s+3)^2+1} \ , \ \mathscr{L}^{-1}\left[\frac{1}{(s+3)^2+1}\right] = e^{-3t}\sin t$$

より

$$x(t) = \mathscr{L}^{-1}[X(s)] = \mathscr{L}^{-1}\left[\frac{1}{(s+3)^2+1} \times F(s)\right] = (e^{-3t}\sin t) * f(t)$$

$$= \int_0^t e^{-3(t-u)}\sin(t-u)f(u)du$$

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問の解答

$\mathscr{L}[x(t)] = X(s)$, $\mathscr{L}[y(t)] = Y(s)$ とおく

$$\mathscr{L}\left[\frac{dx}{dt}\right] = sX(s) - x(0) = sX(s)$$

$$\mathscr{L}\left[\frac{dy}{dt}\right] = sY(s) - y(0) = sY(s) - 1$$

$$\begin{cases} sX(s) = X(s) + Y(s) \\ sY(s) - 1 = -X(s) + 3Y(s) \end{cases} \Rightarrow \begin{array}{ll} (s-1)X(s) - Y(s) = 0 & \cdots ① \\ X(s) + (s-3)Y(s) = 1 & \cdots ② \end{array}$$

① × $(s-3)$ + ② より

$$\begin{array}{rl} (s-3)(s-1)X(s) - (s-3)Y(s) = 0 & \\ X(s) + (s-3)Y(s) = 1 & + \\ \hline \{(s-3)(s-1)+1\}\,X(s) = 1 & \end{array}$$

$$X(s) = \frac{1}{(s-3)(s-1)+1} = \frac{1}{s^2-4s+4} = \frac{1}{(s-2)^2}$$

$$Y(s) = (s-1)X(s) = \frac{s-1}{(s-2)^2} = \frac{1}{s-2} + \frac{1}{(s-2)^2}$$

(答) $x(t) = \mathscr{L}^{-1}[X(s)] = \mathscr{L}^{-1}\left[\frac{1}{(s-2)^2}\right] = te^{2t}$

$$y(t) = \mathscr{L}^{-1}[Y(s)] = \mathscr{L}^{-1}\left[\frac{1}{s-2} + \frac{1}{(s-2)^2}\right] = e^{2t} + te^{2t}$$