< 復元抽出による標本調査 2 >

前のページの定理 14 を証明する。

< (1) の証明>

$$E[X_k] = \sum_{i=1}^{N} y_i P(X_k = y_i) = \frac{1}{N}\sum_{i=1}^{N} y_i = \mu \quad \text{(母平均)}$$

$$E[(X_k-\mu)^2] = \sum_{i=1}^{N}(y_i-\mu)^2 P(X_k = y_i) = \frac{1}{N}\sum_{i=1}^{N}(y_i-\mu)^2 = \sigma^2 \qquad \text{(母分散)}$$

< (2) の証明>

$$E[\overline{X}] = E\left[\frac{1}{n}\sum_{k=1}^{n} X_k\right] = \frac{1}{n}\sum_{k=1}^{n} E[X_k] = \frac{1}{n}\sum_{k=1}^{n}\mu = \mu$$

$$E\left[\sum_{k=1}^{n}(X_k-\overline{X})^2\right] = \sum_{k=1}^{n} E[(X_k-\mu+\mu-\overline{X})^2]$$

$$= \sum_{k=1}^{n}\{E[(X_k-\mu)^2] + 2E[(X_k-\mu)(\mu-\overline{X})] + E[(\mu-\overline{X})^2]\}$$

ここで

$$E[(X_k-\mu)(\mu-\overline{X})] = E\left[(X_k-\mu)\left\{\frac{1}{n}\sum_{i=1}^{n}(\mu-X_i)\right\}\right]$$

$$\frac{1}{n}\sum_{i=1}^{n} E[(X_k-\mu)(\mu-X_i)] = \frac{1}{n}\left\{-E[(X_k-\mu)^2] + \sum_{i\neq k} E[(X_k-\mu)]E[(\mu-X_i)]\right\}$$

$$= -\frac{1}{n}E[(X_k-\mu)^2] = -\frac{\sigma^2}{n}$$

また

$$E[(\mu-\overline{X})^2] = E\left[\left\{\frac{1}{n}\sum_{i=1}^{n}(\mu-X_i)\right\}^2\right]$$

$$= \frac{1}{n^2}\left\{\sum_{i=1}^{n} E[(\mu-X_i)^2] + \sum_{l\neq i} E[(\mu-X_i)(\mu-X_l)]\right\} = \frac{1}{n^2}\left(\sum_{i=1}^{n}\sigma^2\right) = \frac{\sigma^2}{n}$$

となる。よって

$$E\left[\sum_{k=1}^{n}(X_k-\overline{X})^2\right] = \sum_{k=1}^{n}\left\{\sigma^2 + 2\left(-\frac{\sigma^2}{n}\right) + \frac{\sigma^2}{n}\right\} = n\sigma^2\left(1-\frac{1}{n}\right)$$

$$= (n-1)\sigma^2$$

(証明終)