高知工科大学

基礎数学ワークブック

(2004年度版)

初級編

No. 6

解答

＜１ページ. 微分方程式＞

問の解答

(1) $\dfrac{dy}{dt} = 2y$

1 階微分方程式

(2) $\dfrac{d^2y}{dt^2} = -9y$

2 階微分方程式

(3) $\dfrac{d^3y}{dt^3} + \dfrac{d^2y}{dt^2} + t^4 = 0$

3 階微分方程式

< 2ページ. 微分方程式の解 (1) >

問の解答

$y = 3e^t$

($y = -e^t$　など)

＜ 3ページ. 微分方程式の解 (2) ＞

問の解答

(1) $t = 0$ のとき $y = 3$

$C = 3 \qquad \underline{y = 3e^t}$

(2) $t = 0$ のとき $y = -2$

$C = -2 \qquad \underline{y = -2e^t}$

(3) $t = 0$ のとき $y = 0$

$C = 0 \qquad \underline{y = 0}$

< 4ページ. 微分方程式の解 (3) >

問１の解答

$t = 0$ のとき $y = 2$

問２の解答

$y = 3e^{-t}$, $(y = -e^{-t}$など$)$

問３の解答

$y = Ce^{-t}$

＜ 5 ページ. 積分の復習 ＞

問の解答

(1) $\displaystyle\int e^y \frac{dy}{dt} dt = \int e^y dy = e^y + C$

(2) $\displaystyle\int \frac{1}{y^2} \frac{dy}{dt} dt = \int \frac{1}{y^2} dy = -\frac{1}{y} + C$

(3) $\displaystyle\int \sin y \frac{dy}{dt} dt = \int \sin y dy = -\cos y + C$

(4) $\displaystyle\int \cos y \frac{dy}{dt} dt = \int \cos y dy = \sin y + C$

＜ 6 ページ. 求積法 ＞

問の解答

(1) $\dfrac{dy}{dt} = 3t + 6$

$\underline{y = \dfrac{3}{2}t^2 + 6t + C}$

(2) $\dfrac{dy}{dt} = \dfrac{1}{2}t^3 + 5t^4$

$\underline{y = \dfrac{1}{8}t^4 + t^5 + C}$

(3) $\dfrac{dy}{dt} = -\dfrac{2}{t^2} + \dfrac{1}{t}$

$\underline{y = \dfrac{2}{t} + \log|\,t\,| + C}$

(4) $\dfrac{dy}{dt} = 4\sin t - 5\cos t$

$\underline{y = -4\cos t - 5\sin t + C}$

< 7ページ.1階微分方程式の原理 >

問の解答

<証明> $y_1 = e^t$ とおく. $(*)\dfrac{dy}{dt} = y$ の任意の解を y_2 とすると, $(*)$ より

$$y_1{}' = y_1 \quad , \qquad y_2' = y_2$$

である. 今 $y = \dfrac{y_2}{y_1}$ とおくと

$$y' = \frac{y_2{}'y_1 - y_2y_1{}'}{(y_1)^2} = \frac{y_2y_1 - y_2y_1}{(y_1)^2} = 0$$

より定理から y が定数 C になるので

$$y = C \;\Rightarrow\; \frac{y_2}{y_1} = C \;\Rightarrow\; y_2 = Cy_1 = Ce^t$$

より $(*)$ の任意の解 y_2 が $(**)$ の形をしていることがわかった.

(証明終)

< 8 ページ. 変数分離形 (1) >

問の解答

$y = Ce^{2t}$　(C は任意定数)

< 9 ページ. 変数分離形 (2) >

問の解答

(1) $y = Ce^{5t}$　　(C は任意定数)

(2) $y = Ce^{-3t}$　　(C は任意定数)

(3) $y = Ce^{at}$　　(C は任意定数)

< 10 ページ. 変数分離形 (3) >

問の解答

(1) $y = Ce^{3t^2+5t}$　　(C は任意定数)

(2) $y = Ce^{t^3+4t}$　　(C は任意定数)

＜11ページ.1階線形微分方程式 (1)＞

問１の解答

(1) $y = Ce^{-at}$　　(C は任意定数)

(2) $y = Ce^{5t^2}$　　(C は任意定数)

(3) $y = Ce^{-2t^3-t}$　　(C は任意定数)

問２の解答

$y = Ce^{-\int p(t)dt}$　　(C は任意定数)

< 13ページ.1階線形微分方程式 (3) >

問の解答

(1) $y = \dfrac{5}{3} + Ce^{-3t}$　　(C は任意定数)

(2) $y = \dfrac{b}{a} + Ce^{-at}$　　(C は任意定数)

< 14 ページ.1 階線形微分方程式 (4) >

問の解答

(1) $y = C(t)e^{-4t}$

$y' + 4y = C'(t)e^{-4t} = e^{5t}$

$C'(t) = e^{9t}$

$C(t) = \dfrac{1}{9}e^{9t} + C$

(答) $y = \dfrac{1}{9}e^{5t} + Ce^{-4t}$

(2) $y = C(t)e^{4t}$

$y' - 4y = C'(t)e^{4t} = e^{5t}$

$C'(t) = e^{t}$

$C(t) = e^{t} + C$

(答) $y = e^{5t} + Ce^{4t}$

< 15 ページ.1 階線形微分方程式 (5) >

問の解答

(1) $y = te^{2t} + Ce^{2t}$　　(C は任意定数)

(2) $y = te^{-3t} + Ce^{-3t}$　　(C は任意定数)

(3) $y = te^{at} + Ce^{at}$　　(C は任意定数)

< 16 ページ.1 階線形微分方程式 (6) >

問の解答

(1) $\dfrac{dy}{dt} + ay = e^{bt}$

$y = C(t)e^{at}$ とおくと

$\dfrac{dy}{dt} + ay = C'(t)e^{-at} - aC(t)e^{-at} + aC(t)e^{-at} = C'(t)e^{-at} = e^{bt}$ より

$C'(t) = e^{(a+b)t} \ \Rightarrow\ C(t) = \dfrac{1}{a+b}e^{(a+b)t} + C$

よって　$y = \left\{\dfrac{1}{a+b}e^{(a+b)t} + C\right\}e^{-at} = \dfrac{1}{a+b}b^{bt} + Ce^{-at}$

(答)　$y = \dfrac{1}{a+b}e^{bt} + Ce^{-at}$　　(C は任意定数)

(2) $\dfrac{dy}{dt} - ay = e^{bt}$

$y = C(t)e^{at}$ とおくと

$\dfrac{dy}{dt} - ay = C'(t)e^{at} = e^{bt}$ より

$C'(t) = e^{(b-a)t} \ \Rightarrow\ C(t) = \dfrac{1}{b-a}e^{(b-a)t} + C$

よって　$y = \left\{\dfrac{1}{b-a}e^{(b-a)t} + C\right\}e^{at} = \dfrac{1}{b-a}b^{bt} + Ce^{at}$

(答)　$y = \dfrac{1}{b-a}e^{bt} + Ce^{at}$　　(C は任意定数)

(3) $\dfrac{dy}{dt} + ay = e^{-at}$

$y = C(t)e^{-at}$ とおくと

$\dfrac{dy}{dt} + ay = C'(t)e^{-at} = e^{-at}$ より

$C'(t) = 1 \ \Rightarrow\ C(t) = t + C$

よって　$y = \{t + C\}e^{-at} = te^{-at} + Ce^{-at}$

(答)　$y = te^{-at} + Ce^{-at}$　　(C は任意定数)

< 17 ページ.1 階線形微分方程式 (7) >

問の解答

(1) $\dfrac{dx}{dt} + 3x = \cos(4t)$

$x(t) = C(t)e^{-3t}$ とおくと

$$\frac{dx}{dt} + 3x = C'(t)e^{-3t} = \cos(4t) \;\Rightarrow\; C'(t) = e^{3t}\cos(4t)$$

$$C(t) = \int e^{3t}\cos(4t)dt = \frac{e^{3t}}{25}\Big\{3\cos(4t) + 4\sin(4t)\Big\} + C$$

(答) $x(t) = \dfrac{1}{25}\Big\{3\cos(4t) + 4\sin(4t)\Big\} + Ce^{-3t}$　　(C は任意定数)

(2) $\dfrac{dy}{dt} + 3x = \sin(4t)$

$y(t) = C(t)e^{-3t}$ とおくと

$$\frac{dy}{dt} + 3y = C'(t)e^{-3t} = \sin(4t) \;\Rightarrow\; C'(t) = e^{3t}\sin(4t)$$

$$C(t) = \int e^{3t}\sin(4t)dt = \frac{e^{3t}}{25}\Big\{-4\cos(4t) + 3\sin(4t)\Big\} + C$$

(答) $y(t) = \dfrac{1}{25}\Big\{-4\cos(4t) + 3\sin(4t)\Big\} + Ce^{-3t}$　　(C は任意定数)

(3) $\dfrac{dz}{dt} + 3z = e^{4it}$

$z(t) = C(t)e^{-3t}$ とおくと

$$\frac{dz}{dt} + 3z = C'(t)e^{-3t} = e^{4it} \;\Rightarrow\; C'(t) = e^{3t+4it}$$

$$\begin{aligned} C(t) &= \int e^{3t+4it}dt = \frac{1}{3+4i}e^{(3+4i)t} + C \\ &= \frac{3-4i}{3^2+4^2}e^{3t}\Big\{\cos(4t) + i\sin(4t)\Big\} + C \\ &= \frac{e^{3t}}{25}\Big\{3\cos(4t) + 4\sin(4t)\Big\} + i\frac{e^{3t}}{25}\Big\{-4\cos(4t) + 3\sin(4t)\Big\} + C \end{aligned}$$

(答) $z(t) = \dfrac{1}{25}\Big\{3\cos(4t) + 4\sin(4t)\Big\} + i\dfrac{1}{25}\Big\{-4\cos(4t) + 3\sin(4t)\Big\} + Ce^{-3t}$

(C は任意定数)

＜ 18 ページ.1 階線形微分方程式 (8) ＞

問の解答

(1) $y = 2t^3 - 4t^2 + 5t + C$

(2) $y = \dfrac{1}{2}e^{2t} + \dfrac{5}{3}\cos(3t) + 3\sin(2t) + C$

(3) $y = Ce^{10t}$

(4) $y = Ce^{-9t}$

(5) $y = Ce^{t^2+t}$

(6) $y = Ce^{-\frac{t^3}{3}+2t^2}$

(7) $y = \dfrac{3}{4} + Ce^{-4t}$

(8) $y = -\dfrac{5}{3} + Ce^{3t}$

(9) $y = -e^{2t} + Ce^{3t}$

(10) $y = -e^{-3t} + Ce^{-2t}$

(11) $y = te^{3t} + Ce^{3t}$

(12) $y = -\dfrac{1}{6}e^{-3t} + Ce^{3t}$

(13) $y = te^{-3t} + Ce^{-3t}$

< 19 ページ.1 階微分方程式の初期値問題 >

問の解答

(1) $\begin{cases} \dfrac{dy}{dt} = 10 - 9.8t \\ t = 0 \text{ のとき } y = 6 \end{cases}$

$y = 10t - 4.9t^2 + C$

$t = 0$ のとき $y = C = 6$

(答) $y = 10t - 4.9t^2 + 6$

(2) $\begin{cases} \dfrac{dy}{dt} = -5y \\ t = 0 \text{ のとき } y = 4 \end{cases}$

$y = Ce^{-5t}$

$t = 0$ のとき $y = C = 4$

(答) $y = 4e^{-5t}$

(3) $\begin{cases} \dfrac{dy}{dt} + ky = 9.8 \\ t = 0 \text{ のとき } y = 0 \end{cases}$

$y = \dfrac{9.8}{k} + Ce^{-kt}$

$t = 0$ のとき $y = \dfrac{9.8}{k} + C = 0 \ \Rightarrow \ C = -\dfrac{9.8}{k}$

(答) $y = \dfrac{9.8}{k}(1 - e^{-kt})$

(4) $\begin{cases} \dfrac{dy}{dt} + ky = -g \\ t = 0 \text{ のとき } y = 4 \end{cases}$

$y = -\dfrac{g}{k} + Ce^{kt}$

$t = 0$ のとき $y = -\dfrac{g}{k} + C = 4 \ \Rightarrow \ C = \dfrac{g}{k} + 4$

(答) $y = -\dfrac{g}{k}(1 - e^{-kt}) + 4e^{-kt}$

< 20 ページ.2 階線形微分方程式 (1) >

問の解答

(1) $\begin{cases} \dfrac{d^2y}{dt^2} = 8 \\ y(0) = 7\ , \quad y'(0) = 6 \end{cases}$

一般解　$y = 4t^2 + C_1t + C_2$

初期値　$y(0) = 7 \ \Rightarrow\ C_2 = 7 \quad , \qquad y'(0) = 6 \ \Rightarrow\ C_1 = 6$

(答) $y = 4t^2 + 6t + 7$

(2) $\begin{cases} \dfrac{d^2y}{dt^2} = 6t + 2 \\ y(0) = 8\ , \quad y'(0) = 9 \end{cases}$

一般解　$y = t^3 + t^2 + C_1t + C_2$

初期値　$y(0) = 8 \ \Rightarrow\ C_2 = 8 \quad , \qquad y'(0) = 9 \ \Rightarrow\ C_1 = 9$

(答) $y = t^3 + t^2 + 9t + 8$

＜ 21 ページ.2 階線形微分方程式 (2) ＞

問の解答

① $y(t) = C_1 \cos(3t) + C_2 \sin(3t)$,　 $y(0) = C_1 = 6$

$y'(t) = -3C_1 \sin(3t) + 3C_2 \cos(3t)$,　 $y'(0) = 3C_2 = 8$

(答) $y = 6\cos(3t) + \frac{8}{3}\sin(3t)$

② $C_1 = \alpha$,　 $3C_2 = \beta$

(答) $y = \alpha\cos(3t) + \frac{\beta}{3}\sin(3t)$

< 22 ページ.2 階線形同次微分方程式 (1) >

問の解答

もう一つの基本解は $y = \sin(2t)$

一般解は $y = C_1\cos(2t) + C_2\sin(2t)$　　(C_1 , C_2 は任意定数)

< 23 ページ.2 階線形同次微分方程式 (2) >

問の解答

もう一つの基本解は e^{5t}

一般解は $y = C_1 e^{5t} + C_2 t e^{5t}$　　(C_1 , C_2 は任意定数)

< 24 ページ. 微分演算子 D >

問 1 の解答

(1) $(D+5)y=0$

(2) $(D^2-6D+9)y=0$

問 2 の解答

(1) $(D-4)y=0 \;\Rightarrow\; \dfrac{dy}{dt}-4y=0$

(答) $y=Ce^{4t}$　(C は任意定数)

(2) $(D-a)y=0 \;\Rightarrow\; \dfrac{dy}{dt}-ay=0$

(答) $y=Ce^{at}$　(C は任意定数)

(3) $(D-4)y=e^{4t} \;\Rightarrow\; \dfrac{dy}{dt}-4y=e^{4t}$

(答) $y=te^{4t}+Ce^{4t}$　(C は任意定数)

(4) $(D-a)y=e^{at} \;\Rightarrow\; \dfrac{dy}{dt}-ay=e^{at}$

(答) $y=te^{at}+Ce^{at}$　(C は任意定数)

< 25 ページ. 定数係数 2 階線形同次微分方程式 (1) >

問 1 の解答

(1) $y = e^{2t}$,　$\dfrac{d^2y}{dt^2} - 5\dfrac{dy}{dt} + 6y = 4e^{2t} - 5 \times 2e^{2t} + 6 \times e^{2t} = (4 - 10 + 6)e^{2t} = 0$

(2) $y = e^{3t}$,　$\dfrac{d^2y}{dt^2} - 5\dfrac{dt}{dy} + 6y = 9e^{3t} - 5 \times 3e^{3t} + 6 \times e^{3t} = (9 - 15 + 6)e^{3t} = 0$

問 2 の解答

(1) $\dfrac{d^2y}{dy^2} - 3\dfrac{dy}{dt} + 2y = 0$

$\Rightarrow\ (D^2 - 3D + 2)y = 0$

$\Rightarrow\ (D - 1)(D - 2)y = 0$

$\Rightarrow\ \underline{(答)\ y = C_1e^t + C_2e^{2t}}$　(C_1 , C_2 は任意定数)

(2) $\dfrac{d^2y}{dy^2} - 3\dfrac{dy}{dt} - 4y = 0$

$\Rightarrow\ (D^2 - 3D - 4)y = 0$

$\Rightarrow\ (D - 4)(D + 1)y = 0$

$\Rightarrow\ \underline{(答)\ y = C_1e^{4t} + C_2e^{-t}}$　(C_1 , C_2 は任意定数)

< 26 ページ. 定数係数 2 階線形同次微分方程式 (2) >

問の解答

(1) $(D-2)(D-2)y=0$

(答) $y=C_1e^{2t}+C_2te^{2t}$　(C_1 , C_2 は任意定数)

(2) $(D-5)(D-5)y=0$

(答) $y=C_1e^{5t}+C_2te^{5t}$　(C_1 , C_2 は任意定数)

(3) $(D+4)(D+4)y=0$

(答) $y=C_1e^{-4t}+C_2te^{-4t}$　(C_1 , C_2 は任意定数)

(4) $(D-\alpha)(D-\alpha)y=0$

(答) $y=C_1e^{\alpha t}+C_2te^{\alpha t}$　(C_1 , C_2 は任意定数)

＜ 27 ページ. 定数係数 2 階線形同次微分方程式 (3) ＞

問 1 の解答

$$y = \left(\frac{C_1 - C_2 i}{2} + \frac{C_1 + C_2 i}{2}\right)\cos(3t) + i\left(\frac{C_1 - C_2 i}{2} - \frac{C_1 + C_2 i}{2}\right)\sin(3t)$$

$$= C_1\cos(3t) + i(-C_2 i)\sin(3t) = C_1\cos(3t) + C_2\sin(3t)$$

問 2 の解答

(1) $y = C_1\cos(2t) + C_2\sin(2t)$　(C_1 , C_2 は任意の実数定数)

(2) $y = C_1\cos(\omega t) + C_2\sin(\omega t)$　(C_1 , C_2 は任意の実数定数)

＜ 28 ページ. 定数係数 2 階線形同次微分方程式 (4) ＞

問 1 の解答

(1) $y_1 = e^{-2t}\cos(3t)$

$y_1{}' = -2e^{-2t}\cos(3t) - 3e^{-2t}\sin(3t)$

$$\begin{aligned} y_1{}'' &= 4e^{-2t}\cos(3t) + 6e^{-2t}\sin(3t) + 6e^{-2t}\sin(3t) - 9e^{-2t}\cos(3t) \\ &= -5e^{-2t}\cos(3t) + 12e^{-2t}\sin(3t) \end{aligned}$$

$$\frac{d^2y_1}{dt^2} + 4\frac{dy_1}{dt} + 13y_1 = -5e^{-2t}\cos(3t) + 12e^{-2t}\sin(3t) \\ +4\left\{-2e^{-2t}\cos(3t) - 3e^{-2t}\sin(3t)\right\} + 13e^{-2t}\cos(3t) = 0$$

(2) $y_2 = e^{-2t}\sin(3t)$

$y_2{}' = -2e^{-2t}\sin(3t) + 3e^{-2t}\cos(3t)$

$$\begin{aligned} y_2{}'' &= 4e^{-2t}\sin(3t) - 6e^{-2t}\cos(3t) - 6e^{-2t}\cos(3t) - 9e^{-2t}\sin(3t) \\ &= -5e^{-2t}\sin(3t) - 12e^{-2t}\cos(3t) \end{aligned}$$

$$\frac{d^2y_2}{dt^2} + 4\frac{dy_2}{dt} + 13y_2 = 0$$

問 2 の解答

$$\begin{aligned} y &= \left(\frac{C_1 - C_2 i}{2} + \frac{C_1 + C_2 i}{2}\right)e^{-2t}\cos(3t) + i\left(\frac{C_1 - C_2 i}{2} - \frac{C_1 + C_2 i}{2}\right)e^{-2t}\sin(3t) \\ &= C_1 e^{-2t}\cos(3t) + C_2 e^{-2t}\sin(3t) \end{aligned}$$

問 3 の解答

(1) $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 5y = 0$

$D^2 + 4D + 5 = 0$

$D = -2 \pm i$

(答) $y = C_1 e^{-2t}\cos t + C_2 e^{-2t}\sin t$　　(C_1 , C_2 は任意の実数定数)

(2) $\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} + 10y = 0$

$D^2 - 2D + 10 = 0$

$D = 1 \pm 3i$

(答) $y = C_1 e^{t}\cos(3t) + C_2 e^{t}\sin(3t)$　　(C_1 , C_2 は任意の実数定数)

< 29 ページ. 定数係数 2 階線形同次微分方程式 (5) >

問の解答

(1) $D^2 - 5D - 6 = 0$

$(D-6)(D+1) = 0 \;\Rightarrow\; D = 6\,,\ -1$

<u>(答) $y = C_1 e^{6t} + C_2 e^{-t}$</u>　($C_1$, C_2 は任意定数)

(2) $D^2 + 8D + 16 = 0$

$(D+4)(D+4) = 0$

<u>(答) $y = C_1 e^{-4t} + C_2 t e^{-4t}$</u>　($C_1$, C_2 は任意定数)

(3) $(D^2 + 16) = 0$

$D = \pm\sqrt{16}\,i = \pm 4i$

<u>(答) $y = C_1 \cos(4t) + C_2 \sin(4t)$</u>　($C_1$, C_2 は任意定数)

(4) $D^2 - 8D + 20 = 0$

$D = 4 \pm 2i$

<u>(答) $y = C_1 e^{4t} \cos(2t) + C_2 e^{4t} \sin(2t)$</u>　($C_1$, C_2 は任意定数)

< 30 ページ. 定数係数 2 階線形非同次微分方程式 (1) >

問の解答

(1) $D^2 + D - 2 = (D-1)(D+2)$

<u>(答) $y = -3 + C_1 e^{t} + C_2 e^{-2t}$</u>　　$(C_1$, C_2 は任意定数)

(2) $D^2 - 3D - 4 = (D-4)(D+1)$

<u>(答) $y = -2 + C_1 e^{4t} + C_2 e^{-t}$</u>　　$(C_1$, C_2 は任意定数)

(3) $D^2 + 4D + 4 = (D+2)(D+2)$

<u>(答) $y = \dfrac{5}{2} + C_1 e^{-2t} + C_2 t e^{-2t}$</u>　　$(C_1$, C_2 は任意定数)

(4) $D^2 + 16 = (D-4i)(D-4i)$

<u>(答) $y = \dfrac{5}{4} + C_1 \cos(4t) + C_2 \sin(4t)$</u>　　$(C_1$, C_2 は任意定数)

< 31 ページ. 定数係数 2 階線形非同次微分方程式 (2) >

問の解答

$\alpha = 0$,　$\beta = \omega$,　$a = 0$,　$b = \omega^2$

$$\left.\begin{aligned} A &= \alpha^2 - \beta^2 + \alpha a + b = 0 - \omega^2 + 0 + \omega^2 = 0 \\ B &= (2\alpha + a)\beta = 0 \end{aligned}\right) \Rightarrow \text{特解} = -\frac{r}{2\beta} t e^{\alpha t} \cos(\beta t)$$

$$= -\frac{r}{2\omega} t \cos(\omega t)$$

一般解 $\underline{y = C_1 \cos(\omega t) + C_2 \sin(\omega t) - \frac{rt}{2\omega} \cos(\omega t)}$　(C_1 , C_2 は任意定数)

< 32ページ. 定数係数2階線形微分方程式の一般解 >

問の解答

(1) $y = -\dfrac{7}{6}t^3 + 5t^2 + C_1 t + C_2$

(2) $y = C_1 e^{-3t} + C_2 e^{3t}$

(3) $y = C_1 e^{2t} + C_2 e^{-t}$

(4) $y = C_1 e^{t} + C_2 e^{3t}$

(5) $y = C_1 e^{t} + C_2 t e^{t}$

(6) $y = C_1 e^{-3t} + C_2 t e^{-3t}$

(7) $y = C_1 \cos(4t) + C_2 \sin(4t)$

(8) $y = C_1 \cos(5t) + C_2 \sin(5t)$

(9) $y = C_1 e^{-t} \cos t + C_2 e^{-t} \sin t$

(10) $y = C_1 e^{3t} \cos(2t) + C_2 e^{3t} \sin(2t)$

(11) $y = -\dfrac{5}{4} + C_1 e^{2t} + C_2 e^{-2t}$

(12) $y = \dfrac{4}{3} + C_1 e^{t} + C_2 e^{3t}$

(13) $y = \dfrac{2}{3} + C_1 \cos(3t) + C_2 \sin(3t)$

＜ 33 ページ.2 階微分方程式の初期値問題 ＞

問の解答

(1) $D^2 - 3D - 4 = (D-4)(D+1)$

$y = C_1 e^{4t} + C_2 e^{-t}$,　$y(0) = C_1 + C_2 = 5 \cdots$ ①

$y' = 4C_1 e^{4t} - C_2 e^{-t}$,　$y'(0) = 4C_1 - C_2 = 7 \cdots$ ②

①，②より $C_1 = \dfrac{12}{5}$,　$C_2 = \dfrac{13}{5}$

(答) $y(t) = \dfrac{12}{5}e^{4t} + \dfrac{13}{5}e^{-t}$

(2) $D^2 + 4D + 4 = (D+2)(D+2)$

$y = C_1 e^{-2t} + C_2 t e^{-2t}$

$y' = (-2C_1 + C_2)e^{-2t} - 2C_2 t e^{-2t}$

$y(0) = C_1 = 10$,　$y'(0) = -2C_1 + C_2 = 0 \;\Rightarrow\; C_2 = 20$

(答) $y = 10e^{-2t} + 20te^{-2t}$

(3) $D^2 + 25 = (D+5i)(D-5i)$

$y = C_1 \cos(5t) + C_2 \sin(5t)$,　$y(0) = C_1 = 3$

$y' = -5C_1 \sin(5t) + 5C_2 \cos(5t)$,　$y'(0) = 5C_2 = 2$

(答) $y = 3\cos(5t) + \dfrac{2}{5}\sin(5t)$

(4) $D^2 + 4D + 13 = 0$

$D = -2 \pm 3i$

$y = C_1 e^{-2t}\cos(3t) + C_2 e^{-2t}\sin(3t)$

$$y' = -2C_1 e^{-2t}\cos(3t) - 3C_1 e^{-2t}\sin(3t) - 2C_2 e^{-2t}\sin(3t) + 3C_2 e^{-2t}\cos(3t)$$

$$= (-2C_1 + 3C_2)e^{-2t}\cos(3t) - (3C_1 + 2C_2)e^{-2t}\sin(3t)$$

$y(0) = C_1 = 10$,　$y'(0) = -2C_1 + 3C_2 = 0 \;\Rightarrow\; C_2 = \dfrac{20}{3}$

(答) $y = 10e^{-2t}\cos(3t) + \dfrac{20}{3}e^{-2t}\sin(3t)$

< 34 ページ. 微分方程式の練習 (1) >

問の解答

(1) $y = t^3 - 2t^2 + 5t + C$

(2) $y = \dfrac{1}{2}\sin(2t) + C$

(3) $y = Ce^{-5t}$

(4) $y = Ce^{t^2}$

(5) $y = \dfrac{3}{2} + Ce^{-2t}$

(6) $y = -\dfrac{5}{3} + Ce^{3t}$

(7) $y = \dfrac{b}{a} + Ce^{-at}$

(8) $y = -e^t + Ce^{2t}$

(9) $y = te^{3t} + Ce^{3t}$

(10) $y = \dfrac{a}{6}t^3 + \dfrac{b}{2}t^2 + C_1 t + C_2$

(11) $y = C_1 e^{2t} + C_2 e^{3t}$

(12) $y = C_1 e^{5t} + C_2 e^{-t}$

(13) $y = C_1 e^{3t} + C_2 e^{-3t}$

(14) $y = C_1 e^{3t} + C_2 te^{3t}$

(15) $y = C_1 e^{-4t} + C_2 te^{-4t}$

(16) $y = C_1 \cos(3t) + C_2 \sin(3t)$

(17) $y = C_1 \cos(at) + C_2 \sin(at)$

(18) $y = C_1 e^t \cos(2t) + C_2 e^t \sin(2t)$

(19) $y = C_1 e^{-3t} \cos(4t) + C_2 e^{-3t} \sin(4t)$

(20) $y = -\dfrac{2}{3} + C_1 e^{3t} + C_2 e^{-2t}$

(21) $y = \dfrac{5}{4} + C_1 e^{2t} + C_2 te^{2t}$

(22) $y = \dfrac{5}{4} + C_1 \cos(2t) + C_2 \sin(2t)$

< 35 ページ. 微分方程式の練習 (2) >

問の解答

(1) $y = t^2 - 3t + C$

(答) $y = t^2 - 3t + 5$

(2) $y = Ce^{-3t}$

(答) $y = 4e^{-3t}$

(3) $y = \dfrac{5}{4} + Ce^{-4t}$

$y(0) = \dfrac{5}{4} + C = 6$

$\Rightarrow\ C = 6 - \dfrac{5}{4} = \dfrac{24-5}{4} = \dfrac{19}{4}$

(答) $y = \dfrac{5}{4} + \dfrac{19}{4}e^{-4t}$

(4) $v = 2 + Ce^{-3t}$

$v(0) = 2 + C = 5\ \Rightarrow\ C = 3$

(答) $v = 2 + 3e^{-3t}$

(5) $I = \dfrac{5}{2} - Ce^{-2t}$

$I(0) = \dfrac{5}{2} - C = 0\ \Rightarrow\ C = \dfrac{5}{2}$

(答) $I(t) = \dfrac{5}{2} - \dfrac{5}{2}e^{-2t}$

(6) $y = 2t^2 + C_1 t + C_2$,　$C_2 = 1$

$y' = 4t + C_1$,　$C_1 = 3$

(答) $y = 2t^2 + 3t + 1$

(7) $y = C_1 e^{-2t} + C_2 e^{2t}$

$y' = -2C_1 e^{-2t} + 2C_2 e^{2t}$

$y(0) = C_1 + C_2 = 1$

$y'(0) = -2C_1 + 2C_2 = 5$

$C_1 = \dfrac{3}{4}$,　$C_2 = \dfrac{7}{4}$

(答) $y = -\dfrac{3}{4}e^{-2t} + \dfrac{7}{4}e^{2t}$

(8) $y = C_1 e^{t} + C_2 e^{4t}$

$y' = C_1 e^{t} + 4C_2 e^{4t}$

$y(0) = C_1 + C_2 = 2$

$y'(0) = C_1 + 4C_2 = 6$

$C_1 = 2 - \dfrac{4}{3} = \dfrac{2}{3}$,　$3C_2 = 4\ \Rightarrow\ C_2 = \dfrac{4}{3}$

(答) $y = \dfrac{2}{3}e^{t} + \dfrac{4}{3}e^{4t}$

(9) $y = C_1 e^{-2t} + C_2 t e^{-2t}$

$y' = -2C_1 e^{-2t} + C_2 e^{-2t} - 2C_2 t e^{-2t}$

$y(0) = C_1 = 1$

$y'(0) = -2C_1 + C_2 = 0\ \Rightarrow\ C_2 = 2$

(答) $y = e^{-2t} + 2te^{-2t}$

(10) $y = C_1 \cos(2t) + C_2 \sin(2t)$

$y' = -2C_2 \sin(2t) + 2C_2 \cos(2t)$

$y(0) = C_1 = 5$

$y'(0) = 2C_2 = 6\ \Rightarrow\ C_2 = 3$

(答) $y = 5\cos(2t) + 3\sin(2t)$

(11) $y = C_1 \cos(3t) + C_2 \sin(3t)$

$y' = -3C_1 \sin(3t) + 3C_2 \cos(3t)$

$y(0) = C_1 = 1$

$y'(0) = 3C_2 = 0$

(答) $y = \cos(3t)$

(12) $y = C_1 e^{-2t}\cos(3t) + C_2 e^{-2t}\sin(3t)$

$y' = (-2C_1 + 3C_2)e^{-2t}\cos(3t)$
$\quad + (-3C_1 - 2C_2)e^{-2t}\sin(3t)$

$y(0) = C_1 = 1$

$y'(0) = -2C_1 + 3C_2 = 0$

$\Rightarrow\ 3C_2 = 2\ \Rightarrow\ C_2 = \dfrac{2}{3}$

(答) $y = e^{-2t}\cos(3t) + \dfrac{2}{3}e^{-2t}\sin(3t)$

< 36ページ.微分方程式の応用 (1) >

問の解答

$$\frac{dv}{dt} = -9.8$$

t 秒後の速度：$\underline{v(t) = -9.8t + 7}$ (m/s)

$$\frac{dy}{dt} = v(t) = -9.8t + 7$$

t 秒後の高さ：$\underline{y(t) = -4.9t^2 + 7t + 10}$ (m)

＜ 37ページ. 微分方程式の応用 (2) ＞

問1の解答

$$\frac{dv}{dt} + \gamma v = -9.8$$

$$v(t) = -\frac{9.8}{\gamma} + Ce^{-\gamma t}$$

$t = 0$ のとき $v(0) = -\dfrac{9.8}{\gamma} + C = 0 \;\Rightarrow\; C = \dfrac{9.8}{\gamma}$

(答) $\underline{v(t) = -\dfrac{9.8}{\gamma} + \dfrac{9.8}{\gamma} e^{-\gamma t}}$

問2の解答

$$\begin{cases} \dfrac{dv}{dt} = -9.8 - \gamma v \\ t = 0 \text{ のとき } v = 5 \end{cases}$$

$$\Rightarrow \; v(t) = -\frac{9.8}{\gamma} + Ce^{-\gamma t}$$

$t = 0$ のとき $v(0) = -\dfrac{9.8}{\gamma} + C = 5$

$$C = 5 + \frac{9.8}{\gamma}$$

(答) $\begin{cases} \underline{v(t) = -\dfrac{9.8}{\gamma} + \left(5 + \dfrac{9.8}{\gamma}\right) e^{-\gamma t}} \quad \text{(m/s)} \\ \underline{\lim_{t \to \infty} v(t) = -\dfrac{9.8}{\gamma}} \quad \text{(m/s)} \end{cases}$

< 38 ページ. 微分方程式の応用 (3) >

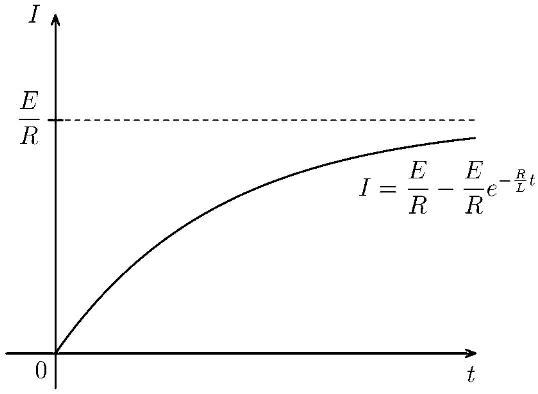
問の解答

$$\frac{dI}{dt}+\frac{R}{L}I=\frac{E}{L}$$

$$I=\frac{\frac{E}{L}}{\frac{R}{L}}+Ce^{-\frac{R}{L}t}=\frac{E}{R}+Ce^{-\frac{R}{L}t}$$

$t=0$ のとき $I=\dfrac{E}{R}+C=0 \;\Rightarrow\; C=-\dfrac{E}{R}$

(答) $I=\dfrac{E}{R}-\dfrac{E}{R}e^{-\frac{R}{L}t}$



< 39 ページ. 微分方程式の応用 (4) >

問の解答

(1) $v = \dfrac{dy}{dt}$ とすると

$$\begin{cases} \dfrac{dv}{dt} + 2v = 0 \\ v(0) = 6 \end{cases} \quad \Rightarrow \quad v(t) = 6e^{-2t}$$

$$y(t) = \int v(t)dt = \int 6e^{-2t}dt = -3e^{-2t} + C$$

$$y(0) = -3 + C = 10 \quad \Rightarrow \quad C = 13$$

<u>(答) $y(t) = -3e^{-2t} + 13$</u>

(2) $v = \dfrac{dy}{dt}$ とすると

$$\begin{cases} \dfrac{dv}{dt} + 2v = 6 \\ v(0) = 8 \end{cases} \quad \Rightarrow \quad \begin{array}{l} v(t) = 3 + C_1e^{-2t} \\ v(0) = 3 + C_1 = 8 \quad \Rightarrow \quad C_1 = 5 \end{array}$$

$$y(t) = \int v(t)dt = \int (3 + 5e^{-2t})dt$$

$$= 3t - \frac{5}{2}e^{-2t} + C_2$$

$$y(0) = -\frac{5}{2} + C_2 = 10 \quad \Rightarrow \quad C_2 = \frac{25}{2}$$

<u>(答) $y(t) = 3t - \dfrac{5}{2}e^{-2t} + \dfrac{25}{2}$</u>

＜ 40 ページ. 微分方程式の応用 (5) ＞

問の解答

(解) t 秒後の速度を $\dfrac{dy}{dt} = v$ とすると

$$\begin{cases} \dfrac{dy}{dt} + \gamma v = -9.8 \\ v(0) = 5 \end{cases}$$

$$\Downarrow$$

$$v = -\frac{9.8}{\gamma} + C_1 e^{-\gamma t}$$

$$v(0) = -\frac{9.8}{\gamma} + C_1 = 5$$

<u>t 秒後の速度は：$v(t) = -\dfrac{9.8}{\gamma} + \left(5 + \dfrac{9.8}{\gamma}\right) e^{-\gamma t}$　(m/s)</u>

$$y(t) = \int v(t)dt = \int \left\{ -\frac{9.8}{\gamma} + \left(5 + \frac{9.8}{\gamma}\right) e^{-\gamma t} \right\} dt$$

$$= -\frac{9.8}{\gamma} t - \frac{1}{\gamma}\left(5 + \frac{9.8}{\gamma}\right) e^{-\gamma t} + C_2$$

$$y(0) = -\frac{1}{\gamma}\left(5 + \frac{9.8}{\gamma}\right) + C_2 = 10$$

$$C_2 = 10 + \frac{1}{\gamma}\left(5 + \frac{9.8}{\gamma}\right)$$

<u>(答) t 秒後の高さは：$y(t) = -\dfrac{9.8}{\gamma} t - \dfrac{1}{\gamma}\left(5 + \dfrac{9.8}{\gamma}\right) e^{-\gamma t} + 10 + \dfrac{1}{\gamma}\left(5 + \dfrac{9.8}{\gamma}\right)$　(m)</u>

< 41 ページ. 微分方程式の応用 (6) >

問の解答

(解) (1) の解は 38 ページより

$$I(t) = \frac{E}{R} - \frac{E}{R}e^{-\frac{R}{L}t} = \frac{dq}{dt}$$

$$q(t) = \int I(t)dt = \int \left(\frac{E}{R} - \frac{E}{R}e^{-\frac{R}{L}t}\right) dt$$

$$= \frac{E}{R}t - \left(-\frac{L}{R}\right)\frac{E}{R}e^{-\frac{R}{L}t} + C$$

$$= \frac{E}{R}t + \frac{LE}{R^2}e^{-\frac{Rt}{L}} + C$$

$$q(0) = \frac{LE}{R^2} + C = 0 \ \Rightarrow \ C = -\frac{LE}{R^2}$$

(答) $q(t) = \frac{E}{R}t + \frac{LE}{R^2}\left(e^{-\frac{Rt}{L}} - 1\right)$

< 42 ページ. ばね (1) >

問の解答

k は大きくなる.

<理由> かたいばねは「伸び ℓ」が小さい. よって

$k = \dfrac{mg}{l}$ は分母が小さくなるので k の値は大きくなる.

< 43 ページ. ばね (2) >

問の解答

$$f = -mg = -kl_0$$

$$\Downarrow$$

$$l_0 = \frac{mg}{k} = \frac{9.8m}{k}$$

＜ 44 ページ. ばねの運動 (1) ＞

問の解答

$$F = m\frac{d^2y}{dt^2} = -ky$$

$$\Rightarrow \quad \frac{d^2y}{dt^2} = -\frac{k}{m}y$$

$$\Rightarrow \quad \underline{\frac{d^2y}{dt^2} + \boxed{\frac{k}{m}}\, y = 0}$$

＜ 45 ページ. ばねの運動 (2) ＞

問 1 の解答

$$\boxed{y(0) = L \quad , \quad y'(0) = 0}$$

問 2 の解答

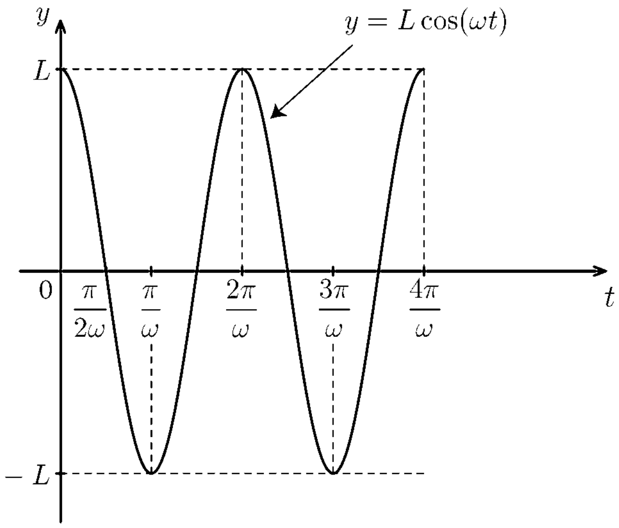
$y(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t) \;\; \Rightarrow \;\; y(0) = C_1 = L$

$y'(t) = -\omega C_1 \sin(\omega t) + \omega C_2 \cos(\omega t) \;\; \Rightarrow \;\; y'(0) = \omega C_2 = 0$

$\omega \neq 0$ より $C_2 = 0$

<u>(答) $y(t) = L\cos(\omega t)$</u>

問 3 の解答



＜ 46 ページ. ばねの運動 (3) ＞

問の解答

(1) $D^2 + 4D + 13 = 0 \ \Rightarrow\ D = -2 \pm 3i$

$y = C_1 e^{-2t}\cos(3t) + C_2 e^{-2t}\sin(3t) \ \Rightarrow\ y(0) = C_1 = L$

$y' = (-2C_1 + 3C_2)e^{-2t}\cos(3t) + (-3C_1 - 2C_2)e^{-2t}\sin(3t)$

$\Rightarrow\ y'(0) = -2C_1 + 3C_2 = 0$

$3C_2 = 2C_1 = 2L$

(答) $y = Le^{-2t}\cos(3t) + \dfrac{2}{3}Le^{-2t}\sin(3t)$

(2) $D^2 + 6D + 9 = (D+3)(D+3)$

$y = C_1 e^{-3t} + C_2 te^{-3t} \ \Rightarrow\ y(0) = C_1 = L$

$y' = (-3C_1 + C_2)e^{-3t} + 3C_2 te^{-3t} \ \Rightarrow\ y'(0) = -3C_1 + C_2 = 0$

(答) $y = Le^{-3t} + 3Lte^{-3t}$

< 47 ページ. 強制振動 (1) >

問 1 の解答

$y = C_1 \cos(5t) + C_2 \sin(5t) + \dfrac{1}{9}\sin(4t)$　　(C_1 , C_2 は任意定数)

問 2 の解答

$y(0) = 0$,　$y'(0) = 0$

問 3 の解答

$y(0) = C_1 = 0$

$y' = -5C_1 \sin(5t) + 5C_2 \cos(5t) + \dfrac{4}{9}\cos(4t)$

$\Rightarrow \quad y'(0) = 5C_2 + \dfrac{4}{9} = 0$

$C_2 = \dfrac{4}{45}$

$y = -\dfrac{4}{45}\sin(5t) + \dfrac{1}{9}\sin(4t)$

問 4 の解答

$y = C_1 \cos(5t) + C_2 \sin(5t) + \dfrac{1}{25-\beta^2}\sin(\beta t) \quad \Rightarrow \quad y(0) = C_1 = 0$

$y' = -5C_1 \sin(5t) + 5C_2 \cos(5t) + \dfrac{\beta}{25-\beta^2}\cos(\beta t) \quad \Rightarrow \quad y'(0) = 5C_2 + \dfrac{\beta}{25-\beta^2} = 0$

$y = -\dfrac{\beta}{5(25-\beta^2)}\sin(5t) + \dfrac{1}{25-\beta^2}\sin(\beta t)$

< 48 ページ. 強制振動 (2) >

問 1 の解答

$$\underline{y = C_1\cos(5t) + C_2\sin(5t) - \frac{t}{10}\cos(5t)} \quad (C_1\ ,\ C_2 \text{ は任意定数})$$

問 2 の解答

$$y(0) = C_1 = 0$$

$$y' = -5C_1\sin(5t) + 5C_2\cos(5t) - \frac{1}{10}\cos(5t) + \frac{t}{2}\sin(5t) \ \Rightarrow\ y'(0) = 5C_2 - \frac{1}{10} = 0$$

$$\underline{y = \frac{1}{50}\sin(5t) - \frac{t}{10}\cos(5t)}$$

問 3 の解答

$$\lim_{\beta\to5}\left\{-\frac{\beta}{5(25-\beta^2)}\sin(5t) + \frac{1}{25-\beta^2}\sin(\beta t)\right\} = \lim_{\beta\to5}\frac{-\beta\sin(5t) + 5\sin(\beta t)}{125-5\beta^2}$$

$$= \lim_{\beta\to5}\frac{\frac{d}{d\beta}\left\{-\beta\sin(5t) + 5\sin(\beta t)\right\}}{\frac{d}{d\beta}(125-5\beta^2)} = \lim_{\beta\to5}\frac{-\sin(5t) + 5t\cos(\beta t)}{-10\beta} = \frac{-\sin(5t) + 5t\cos(5t)}{-50}$$

$$\underline{= \frac{1}{50}\sin(5t) - \frac{t}{10}\cos(5t)}$$

< 49 ページ. まとめの問題 (1) >

問の解答

(1) $y = t^2 - 3t + C$

(2) $y = t^4 + 2t^3 + 4t^2 - 5t + C$

(3) $y = Ce^{4t}$

(4) $y = Ce^{-3t}$

(5) $y = Ce^{3t^2}$

(6) $y = Ce^{t^2+t}$

(7) $y = \dfrac{4}{3} + Ce^{-3t}$

(8) $y = -3 + Ce^{2t}$

(9) $y = \dfrac{1}{9}e^{5t} + Ce^{-4t}$

(10) $y = e^{3t} + Ce^{2t}$

(11) $y = te^{2t} + Ce^{2t}$

(12) $y = te^{-3t} + Ce^{-3t}$

(13) $y = -4.9t^2 + C_1t + C_2$

(14) $y = t^3 + t^2 + C_1t + C_2$

(15) $y = C_1e^{t} + C_2e^{2t}$

(16) $y = C_1e^{5t} + C_2e^{-t}$

(17) $y = C_1e^{2t} + C_2te^{2t}$

(18) $y = C_1e^{-5t} + C_2te^{-5t}$

(19) $y = C_1\cos(2t) + C_2\sin(2t)$

(20) $y = C_1\cos(3t) + C_2\sin(3t)$

(21) $y = C_1e^{2t}\cos t + C_2e^{2t}\sin t$

(22) $y = C_1e^{-t}\cos(\sqrt{3}t) + C_2e^{-t}\sin(\sqrt{3}t)$

(23) $y = -\dfrac{5}{4} + C_1e^{2t} + C_2e^{-2t}$

(24) $y = -3 + C_1e^{2t} + C_2e^{-t}$

< 50 ページ. まとめの問題 (2) >

問 1 の解答

(1) $y = 4e^{5t}$　　　(2) $y = 2 + 3e^{-4t}$

(3) $y = 2e^{t} + e^{2t}$　　　(4) $y = Le^{-3t}\cos(2t) + \dfrac{3}{2}Le^{-3t}\sin(2t)$

問 2 の解答

初期条件は $y(0) = 10$, $y'(0) = 6$ である。 $v(t) = \frac{dy}{dt} = y'(t)$ とおくと

$$\begin{cases} \dfrac{dv}{dt} = -g - \gamma v \\ v(0) = 6 \end{cases} \quad \text{より} \quad \underline{v(t) = -\frac{g}{\gamma} + \left(6 + \frac{g}{\gamma}\right)e^{-\gamma t} \quad (\mathrm{m/s})}$$

$$\begin{cases} y(t) = \int v(t)\ dt \\ y(0) = 10 \end{cases} \quad \text{より} \quad \underline{(\text{答})\ y(t) = -\frac{g}{\gamma}t + \frac{1}{\gamma}\left(6 + \frac{g}{\gamma}\right)\left(1 - e^{-\gamma t}\right) + 10 \quad (\mathrm{m})}$$

問 3 の解答

(1)　$z(t) = C(t)e^{-at}$ とおくと

$$\frac{dz}{dt} + az = C'(t)e^{-at} = e^{bit} \ \Rightarrow\ C'(t) = e^{(a+bi)t}$$

$$\Rightarrow\ C(t) = \int e^{(a+bi)t}dt = \frac{1}{a+bi}e^{(a+bi)t} + C$$

$$= \frac{a-bi}{a^2+b^2}e^{at}\Big\{\cos(bt) + i\sin(bt)\Big\} + C$$

$$= \frac{e^{at}}{a^2+b^2}\Big\{\Big(a\cos(bt) + b\sin(bt)\Big) + i\Big(-b\cos(bt) + a\sin(bt)\Big)\Big\} + C$$

$$\underline{(\text{答})\quad z(t) = \left(\frac{a\cos(bt) + b\sin(bt)}{a^2+b^2}\right) + i\left(\frac{-b\cos(bt) + a\sin(bt)}{a^2+b^2}\right) + Ce^{-at}}$$

(2)

$$(\text{答})\quad \underline{x(t) = \frac{a\cos(bt) + b\sin(bt)}{a^2+b^2} + Ce^{-at}}\ ,\ \underline{y(t) = \frac{-b\cos(bt) + a\sin(bt)}{a^2+b^2} + Ce^{-at}}$$