

高知工科大学

基礎数学ワークブック

(2004年度版)

初級編

No. 4

解答

< 1 ページ. 数列の類推 >

問 1 の解答

$$a_1 = 1 \quad a_2 = 5 \quad a_3 = 14 \quad a_4 = 30 \quad a_5 = 55$$

$$b_1 = 2 \quad b_2 = 3 \quad b_3 = 4 \quad b_4 = 5 \quad b_5 = 6$$

$$b_n = n + 1$$

問 2 の解答

$$a_1 = 1 \quad a_2 = 3 \quad a_3 = 6 \quad a_4 = 10 \quad a_5 = 15$$

$$b_1 = 1 \quad b_2 = 9 \quad b_3 = 36 \quad b_4 = 100 \quad b_5 = 225$$

$$b_n = (a_n)^2$$

< 2 ページ. 和の記号 \sum (1) >

問 1 の解答

$$(1) \sum_{k=1}^7 k = 1 + 2 + 3 + 4 + 5 + 6 + 7$$

$$(2) \sum_{k=1}^5 2^k = 2 + 4 + 8 + 16 + 32$$

$$(3) \sum_{k=1}^6 k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2$$

$$(4) \sum_{k=1}^4 (3k - 1) = 2 + 5 + 8 + 11$$

問 2 の解答

$$(1) 1 + 2 + 3 + 4 + \cdots + n = \sum_{k=1}^n k$$

$$(2) 1 \times 2 + 3 \times 4 + 5 \times 6 + \cdots + (2n - 1)(2n) = \sum_{k=1}^n (2k - 1)(2k)$$

$$(3) 5 + 10 + 15 + \cdots + 100 = \sum_{k=1}^{20} 5k$$

問 3 の解答

$$(1) \sum_{k=3}^7 (k^2 - 8) = (9 - 8) + (16 - 8) + (25 - 8) + (36 - 8) + (49 - 8)$$

$$(2) \sum_{k=4}^8 (2k - 1)(2k + 1) = 7 \times 9 + 9 \times 11 + 11 \times 13 + 13 \times 15 + 15 \times 17$$

$$(3) \sum_{k=0}^5 4^k = 1 + 4 + 4^2 + 4^3 + 4^4 + 4^5$$

< 3 ページ. 和の記号 \sum (2) >

問 1 の解答

$$(1) \sum_{k=1}^n (2k + 3) = 2 \sum_{k=1}^n k + 3 \sum_{k=1}^n 1 = 2 \times \frac{n(n+1)}{2} + 3 \times n = n^2 + 4n$$

$$(2) \sum_{k=1}^n (8k - 5) = 8 \times \frac{n(n+1)}{2} - 5 \times n = 4n^2 + 4n - 5n = 4n^2 - n$$

問 2 の解答

$$(1) \sum_{k=1}^n (2k - 1) = 2 \times \frac{n(n+1)}{2} - n = n^2$$

$$(2) \sum_{k=1}^n (5k - 3) = 5 \times \frac{n(n+1)}{2} - 3n = \frac{5}{2}n^2 - \frac{n}{2}$$

$$(3) \sum_{k=1}^n (7k - 4) = 7 \times \frac{n(n+1)}{2} - 4n = \frac{7}{2}n^2 - \frac{1}{2}n$$

< 4 ページ. 和の記号 \sum (3) >

問 1 の解答

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

問 2 の解答

$$(1) \sum_{k=1}^7 k^2 = \frac{7 \times (7+1) \times (2 \times 7 + 1)}{6} = 140$$

$$(2) \sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

問 3 の解答

$$\begin{aligned} (1) \sum_{k=1}^n (5 - 2k + 12k^2) &= 5n - 2 \times \frac{n(n+1)}{2} + 12 \times \frac{n(n+1)(2n+1)}{6} \\ &= 5n - n(n+1) + 2n(n+1)(2n+1) \\ &= n(4n^2 + 5n + 6) \end{aligned}$$

$$\begin{aligned} (2) \sum_{k=1}^n (3 + 2k)^2 &= \sum_{k=1}^n (9 + 12k + 4k^2) \\ &= 9n + 12 \left(\frac{n(n+1)}{2} \right) + 4 \left(\frac{n(n+1)(2n+1)}{6} \right) \\ &= 9n + 6n(n+1) + \frac{2}{3}n(n+1)(2n+1) \end{aligned}$$

< 5 ページ. 和の記号 \sum (4) >

問 1 の解答

$$\sum_{k=1}^n k^3 = \left\{ \frac{n(n+1)}{2} \right\}^2$$

問 2 の解答

$$(1) \sum_{k=1}^7 k^3 = \left(\frac{7 \times 8}{2} \right)^2 = 784$$

$$(2) \sum_{k=1}^{n-1} k^3 = \left\{ \frac{(n-1)n}{2} \right\}^2$$

問 3 の解答

$$\begin{aligned} (1) \sum_{k=1}^n (1-k)^3 &= \sum_{k=1}^n (1-3k+3k^2-k^3) \\ &= n - \frac{3}{2}n(n+1) + \frac{n(n+1)(2n+1)}{2} - \frac{n^2(n+1)^2}{4} \end{aligned}$$

$$\begin{aligned} (2) \sum_{k=1}^n (2+k)^3 &= \sum_{k=1}^n (8+12k+6k^2+k^3) \\ &= 8n + 6n(n+1) + n(n+1)(2n+1) + \frac{n^2(n+1)^2}{4} \end{aligned}$$

< 6 ページ. 極限 >

問の解答

$$(1) \lim_{n \rightarrow \infty} \frac{3}{2n+4} = 0$$

$$(2) \lim_{n \rightarrow \infty} \frac{3n}{2n-1} = \frac{3}{2}$$

$$(3) \lim_{n \rightarrow \infty} \frac{2n^2 + 3n - 5}{3n^2 - 2n + 4} = \frac{2}{3}$$

$$(4) \lim_{n \rightarrow \infty} \frac{(2n-3)(n+2)}{(n-1)(n+1)} = \lim_{n \rightarrow \infty} \frac{2n^2 + n - 6}{n^2 - 1} = 2$$

$$(5) \lim_{n \rightarrow \infty} \frac{1}{n^3} (1^2 + 2^2 + 3^2 + \cdots + n^2) = \lim_{n \rightarrow \infty} \frac{1}{n^3} \times \frac{n(n+1)(2n+1)}{6} = \frac{1}{3}$$

$$(6) \lim_{n \rightarrow \infty} \frac{1}{n^4} (1^3 + 2^3 + 3^3 + \cdots + n^3) = \lim_{n \rightarrow \infty} \frac{1}{n^4} \times \frac{n^2(n+1)^2}{4} = \frac{1}{4}$$

< 7ページ. 和の極限值 >

問の解答

$$(1) \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n 1 = \lim_{n \rightarrow \infty} \frac{1}{n} \times n = 1$$

$$(2) \lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k=1}^n k = \lim_{n \rightarrow \infty} \frac{1}{n^2} \times \frac{n(n+1)}{2} = \frac{1}{2}$$

$$(3) \lim_{n \rightarrow \infty} \frac{1}{n^3} \sum_{k=1}^n k^2 = \lim_{n \rightarrow \infty} \frac{1}{n^3} \times \frac{n(n+1)(2n+1)}{6} = \frac{1}{3}$$

$$(4) \lim_{n \rightarrow \infty} \frac{1}{n^4} \sum_{k=1}^n k^3 = \lim_{n \rightarrow \infty} \frac{1}{n^4} \times \frac{n^2(n+1)^2}{4} = \frac{1}{4}$$

$$(5) \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \left(1 + \frac{k}{n}\right) = \lim_{n \rightarrow \infty} \frac{1}{n} \left\{ n + \frac{1}{n} \times \frac{n(n+1)}{2} \right\} = \frac{3}{2}$$

$$(6) \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \left(4 + \frac{4k}{n} + \frac{k^2}{n^2}\right) = \lim_{n \rightarrow \infty} \frac{1}{n} \left\{ 4n + 2(n+1) + \frac{(n+1)(2n+1)}{6n} \right\} = \frac{19}{3}$$

$$(7) \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \left(1 + \frac{k}{n}\right)^3 = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \left(1 + \frac{3k}{n} + \frac{3k^2}{n^2} + \frac{k^3}{n^3}\right)$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \left\{ n + \frac{3(n+1)}{2} + \frac{(n+1)(2n+1)}{2n} + \frac{(n+1)^2}{4n} \right\}$$

$$= \frac{15}{4}$$

< 9 ページ. 区分求積法 (2) >

問 1 の解答

$$\sum_{k=1}^{n-1} k^2 = \boxed{\frac{(n-1)n(2n-1)}{6}}$$

$$S_n = \left\{ \frac{1}{6}(n-1)n(2n-1) \right\} \left(\frac{1}{n} \right)^3 = \frac{1}{6} \left(1 - \boxed{\frac{1}{n}} \right) \left(2 - \boxed{\frac{1}{n}} \right)$$

問 2 の解答

$$S_1 = \frac{1}{6} \times (1-1)(2-1) = 0$$

$$S_2 = \frac{1}{6} \times \left(1 - \frac{1}{2} \right) \times \left(2 - \frac{1}{2} \right) = \frac{1}{6} \times \frac{1}{2} \times \frac{3}{2} = \frac{1}{8}$$

$$S_3 = \frac{1}{6} \times \left(1 - \frac{1}{3} \right) \times \left(2 - \frac{1}{3} \right) = \frac{1}{6} \times \frac{2}{3} \times \frac{5}{3} = \frac{5}{27}$$

問 3 の解答

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{1}{6} \left(1 - \frac{1}{n} \right) \left(2 - \frac{1}{n} \right) = \frac{1}{6} \times 1 \times 2 = \frac{1}{3}$$

< 10 ページ. 区分求積法 (3) >

問の解答

$$(1) S_n^* = x_1^2 \Delta x + x_2^2 \Delta x + \cdots + x_n^2 \Delta x = (x_1^2 + x_2^2 + \cdots + x_n^2) \Delta x$$

$$\begin{aligned} (2) S_n^* &= ((\Delta x)^2 + (2\Delta x)^2 + \cdots + (n\Delta x)^2) \Delta x \\ &= (1^2 + 2^2 + \cdots + n^2) (\Delta x)^3 \\ &= \left(\sum_{k=1}^n k^2 \right) \times (\Delta x)^3 \end{aligned}$$

$$(3) S_n^* = \frac{n(n+1)(2n+1)}{6} \times \left(\frac{1}{n} \right)^3 = \frac{(n+1)(2n+1)}{6n^2}$$

$$(4) S_1^* = \frac{1 \times 3}{6} = 1 \quad S_2^* = \frac{3 \times 5}{6 \times 4} = \frac{5}{8} \quad S_3^* = \frac{4 \times 7}{6 \times 9} = \frac{14}{27}$$

$$(5) \lim_{n \rightarrow \infty} S_n^* = \lim_{n \rightarrow \infty} \frac{(n+1)(2n+1)}{6n^2} = \lim_{n \rightarrow \infty} \frac{2n^2 + 3n + 1}{6n^2} = \frac{1}{3}$$

$$(6) S = \frac{1}{3}$$

< 11 ページ. 区分求積法 (4) >

問 1 の解答

$$\begin{aligned}(1) \quad S_n &= (\Delta x)^3 \Delta x + (2\Delta x)^3 \Delta x + \cdots + ((n-1)\Delta x)^3 \Delta x \\ &= \{1 + 2^3 + 3^3 + \cdots + (n-1)^3\} (\Delta x)^4 \\ &= \left\{ \sum_{k=1}^{n-1} k^3 \right\} (\Delta x)^4\end{aligned}$$

$$(2) \quad S_n = \frac{(n-1)^2 n^2}{2^2} \times \frac{1}{n^4} = \frac{(n-1)^2}{4n^2}$$

$$(3) \quad \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{n^2 - 2n + 1}{4n^2} = \frac{1}{4}$$

問 2 の解答

$$S_n^* = \left\{ \sum_{k=1}^n k^3 \right\} (\Delta x)^4 = \left\{ \frac{n(n+1)}{2} \right\}^2 \times \left(\frac{1}{n} \right)^4 = \frac{n^2 + 2n + 1}{4n^2}$$

$$\lim_{n \rightarrow \infty} S_n^* = \lim_{n \rightarrow \infty} \frac{n^2 + 2n + 1}{4n^2} = \frac{1}{4}$$

問 3 の解答

$$S = \frac{1}{4}$$

< 12 ページ. 区分求積法 (5) >

問 1 の解答

$$(1) S_n(x) = (x_1^2 + x_2^2 + \cdots + x_{n-1}^2) \Delta x$$

$$(2) S_n(x) = \{ \Delta x^2 + (2\Delta x)^2 + \cdots + ((n-1)\Delta x)^2 \} \Delta x = \left\{ \sum_{k=1}^{n-1} k^2 \right\} (\Delta x)^3$$

$$(3) S_n(x) = \frac{(n-1)n(2n-1)}{6} \times \frac{x^3}{n^3} = \frac{(n-1)(2n-1)}{6n^2} x^3$$

$$(4) \lim_{n \rightarrow \infty} S_n(x) = \lim_{n \rightarrow \infty} \left(\frac{2n^2 - 3n + 1}{6n^2} \right) x^3 = \frac{1}{3} x^3$$

問 2 の解答

$$(1) S_n^*(x) = \sum_{k=1}^n x_k^2 \Delta x = \sum_{k=1}^n k^2 (\Delta x)^3 = \frac{(n+1)(2n+1)}{6n^2} x^3$$

$$(2) \lim_{n \rightarrow \infty} S_n^*(x) = \lim_{n \rightarrow \infty} \frac{2n^2 + 3n + 1}{6n^2} x^3 = \frac{1}{3} x^3$$

問 3 の解答

$$S(x) = \frac{1}{3} x^3$$

< 13 ページ. 区分求積法 (6) >

問 1 の解答

$$(1) S_n(x) = (x_1^3 + x_2^3 + \cdots + x_{n-1}^3) \Delta x$$

$$(2) S_n(x) = \sum_{k=1}^{n-1} x_k^3 \Delta x = \left\{ \sum_{k=1}^{n-1} k^3 \right\} (\Delta x)^4$$

$$(3) S_n(x) = \left\{ \frac{(n-1)n}{2} \right\}^2 \times \left(\frac{x}{n} \right)^4 = \frac{(n-1)^2}{2^2 n^2} x^4$$

$$(4) \lim_{n \rightarrow \infty} S_n(x) = \lim_{n \rightarrow \infty} \frac{n^2 - 2n + 1}{4n^2} x^4 = \frac{1}{4} x^4$$

問 2 の解答

$$(1) S_n^*(x) = \left\{ \sum_{k=1}^n k^3 \right\} (\Delta x)^4 = \frac{(n+1)^2}{4n^2} x^4$$

$$(2) \lim_{n \rightarrow \infty} S_n^*(x) = \lim_{n \rightarrow \infty} \frac{n^2 + 2n + 1}{4n^2} x^4 = \frac{1}{4} x^4$$

問 3 の解答

$$S(x) = \frac{1}{4} x^4$$

< 14 ページ. 面積関数 (1) >

問 1 の解答

(1) $S(x) = x$

(2) $S(x) = \frac{1}{2}x^2$

問 2 の解答

(1) $S(x) = \frac{1}{3}x^3$

(2) $S(x) = \frac{1}{4}x^4$

問 3 の解答

(1) $S(x) = \frac{1}{5}x^5$

(2) $S(x) = \frac{1}{n+1}x^{n+1}$

問 4 の解答

$$S'(x) = f(x) \quad \left(\int f(x)dx = S(x) \right)$$

< 15 ページ. 面積関数 (2) >

問の解答

[証明]

$$f(x + \Delta x)\Delta x \leq S(x + \Delta x) - S(x) \leq f(x)\Delta x$$

より

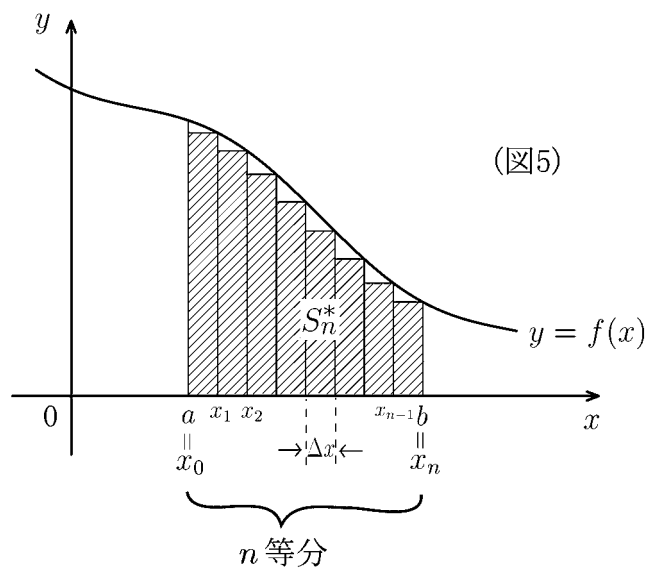
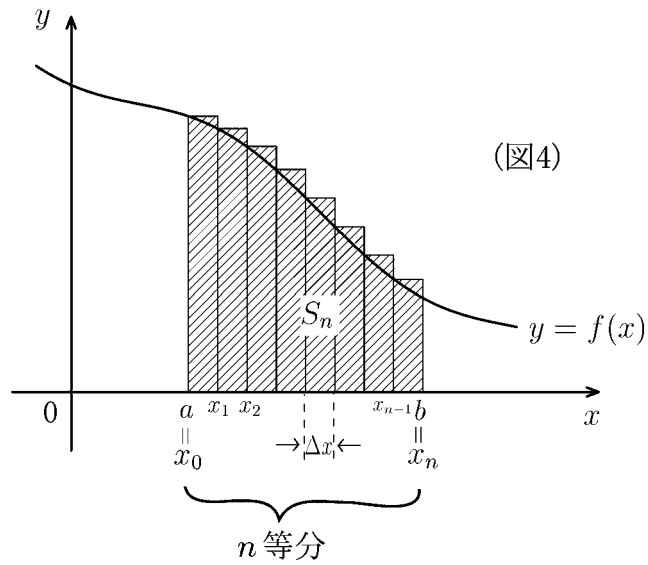
$$\lim_{\Delta x \rightarrow 0} f(x + \Delta x) \leq \lim_{\Delta x \rightarrow 0} \frac{S(x + \Delta x) - S(x)}{\Delta x} \leq f(x)$$

よって $f(x) \leq S'(x) \leq f(x)$ より $S'(x) = f(x)$ が成立する。

(証明終)

< 16 ページ. 定積分の定義 >

問の解答



< 17 ページ. 和の極限の定積分表示 (1) >

問の解答

$$(1) \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(1 + \frac{k}{n}\right) \frac{1}{n} = \int_1^2 x \, dx \quad \left(= \int_0^1 (1+x) \, dx\right)$$

$$(2) \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(2 + \frac{k}{n}\right)^2 \frac{1}{n} = \int_2^3 x^2 \, dx \quad \left(= \int_0^1 (2+x)^2 \, dx\right)$$

$$(3) \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(1 + \frac{k}{n}\right)^3 \frac{1}{n} = \int_1^2 x^3 \, dx \quad \left(= \int_0^1 (1+x)^3 \, dx\right)$$

$$(4) \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(3 + \frac{2}{n}k\right)^4 \frac{2}{n} = \int_3^5 x^4 \, dx \quad \left(= \int_0^2 (3+x)^4 \, dx\right)$$

< 18 ページ. 和の極限の定積分表示 (2) >

問の解答

$$(1) \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{2 + \frac{k}{n}} \right) \frac{1}{n} = \int_2^3 \frac{1}{x} dx \quad \left(= \int_0^1 \frac{1}{2+x} dx \right)$$

$$(2) \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{\left(4 + \frac{2k}{n}\right)^5} \right) \frac{2}{n} = \int_4^6 \frac{1}{x^5} dx \quad \left(= \int_0^2 \frac{1}{(4+x)^5} dx \right)$$

$$(3) \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{\sqrt{1 + \frac{3k}{n}}} \right) \frac{3}{n} = \int_1^4 \frac{1}{\sqrt{x}} dx \quad \left(= \int_0^3 \frac{1}{\sqrt{1+x}} dx \right)$$

$$(4) \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \sqrt[3]{2 + \frac{4k}{n}} \right) \frac{4}{n} = \int_2^6 \sqrt[3]{x} dx \quad \left(= \int_0^4 \sqrt[3]{2+x} dx \right)$$

< 20 ページ. 定積分 (1) >

問の解答

(1)
$$\int_a^b x^n dx = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}$$

(2)
$$\int_a^b \frac{1}{x} dx = \log \left| \frac{b}{a} \right|$$

(3)
$$\int_a^b dx = b - a$$

(4)
$$\int_a^b e^x dx = e^b - e^a$$

(5)
$$\int_a^b \cos x dx = \sin b - \sin a$$

(6)
$$\int_a^b \sin x dx = -\cos b + \cos a$$

(7)
$$\int_a^b \frac{dx}{\cos^2 x} = \tan b - \tan a$$

(8)
$$\int_a^b \frac{dx}{1+x^2} = \tan^{-1} b - \tan^{-1} a$$

(9)
$$\int_4^{10} dx = 6$$

(10)
$$\int_1^5 \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_1^5 = \frac{4}{5}$$

(11)
$$\int_1^4 \sqrt{x} dx = \left[\frac{2}{3} x \sqrt{x} \right]_1^4 = \frac{14}{3}$$

(12)
$$\int_1^8 \frac{1}{\sqrt[3]{x}} dx = \left[\frac{3}{2} \sqrt[3]{x^2} \right]_1^8 = \frac{9}{2}$$

(13)
$$\int_1^e \frac{1}{x} dx = \log e - \log 1 = 1$$

(14)
$$\int_0^2 e^x dx = e^2 - 1$$

(15)
$$\int_0^{\frac{\pi}{2}} \cos x dx = \sin \frac{\pi}{2} - \sin 0 = 1$$

(16)
$$\int_0^{\pi} \sin x dx = -\cos \pi + \cos 0 = 2$$

(17)
$$\int_0^{\frac{\pi}{4}} \frac{dx}{\cos^2 x} = \tan \frac{\pi}{4} - \tan 0 = 1$$

(18)
$$\int_0^1 \frac{dx}{1+x^2} = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4}$$

< 21 ページ. 定積分 (2) >

問の解答

$$\begin{aligned}
 (1) \int_1^2 \frac{2x^2 - 3x + 1}{x^2} dx &= \int_1^2 \left(2 - \frac{3}{x} + \frac{1}{x^2} \right) dx \\
 &= \left[2x - 3 \log |x| - \frac{1}{x} \right]_1^2 \\
 &= 4 - 3 \log 2 - \frac{1}{2} - (2 - 1) \\
 &= \frac{5}{2} - 3 \log 2
 \end{aligned}$$

$$(2) \int_{-1}^0 (x^3 + x^4) dx + \int_0^1 (x^3 + x^4) dx = \int_{-1}^1 (x^3 + x^4) dx = \left[\frac{1}{4} x^4 + \frac{1}{5} x^5 \right]_{-1}^1 = \frac{2}{5}$$

$$(3) \int_1^2 \frac{dx}{x(x+1)} = \int_1^2 \left(\frac{1}{x} - \frac{1}{x+1} \right) dx = \left[\log \left| \frac{x}{x+1} \right| \right]_1^2 = \log \frac{2}{3} - \log \frac{1}{2} = \log \frac{4}{3}$$

$$(4) \int_0^\pi \cos^2 x dx = \int_0^\pi \frac{1 + \cos(2x)}{2} dx = \left[\frac{x}{2} + \frac{\sin(2x)}{4} \right]_0^\pi = \frac{\pi}{2}$$

$$\begin{aligned}
 (5) \int_0^{\frac{\pi}{2}} \sin(2x) \cos(4x) dx &= \int_0^{\frac{\pi}{2}} \frac{\sin(6x) + \sin(-2x)}{2} dx \\
 &= \left[\frac{-\cos(6x)}{12} + \frac{\cos(-2x)}{4} \right]_0^{\frac{\pi}{2}} \\
 &= -\frac{\cos(3\pi)}{12} + \frac{\cos(-\pi)}{4} - \left(\frac{-1}{12} + \frac{1}{4} \right) \\
 &= -\frac{1}{3}
 \end{aligned}$$

< 22 ページ. 定積分 (3) >

問の解答

$$(1) \int_{-1}^1 (x^3 + x^4 + x^5) dx = 2 \int_0^1 x^4 dx = \frac{2}{5}$$

$$(2) \int_{-1}^1 (x + x^3 + x^6) dx = 2 \int_0^1 x^6 dx = \frac{2}{7}$$

$$(3) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin x + \cos x) dx = 2 \int_0^{\pi} \cos x dx = 2$$

$$(4) \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{1}{\cos^2 x} + \tan x \right) dx = 2 \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x} dx = 2$$

< 23 ページ. 定積分の積分変数 >

問の解答

$$(1) \int_1^3 (4 - 9.8t) dt = [4t - 4.9t^2]_1^3 = (12 - 4.9 \times 9) - (4 - 4.9) = -31.2$$

$$(2) \int_0^R 2\pi r dr = [\pi r^2]_0^R = \pi R^2$$

$$(3) \int_0^\pi \sin \theta d\theta = [-\cos \theta]_0^\pi = -\cos \pi + \cos 0 = 2$$

$$(4) \int_a^b u^n du = \left[\frac{1}{n+1} u^{n+1} \right]_a^b = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}$$

$$(5) \int_1^9 \sqrt{u} du = \left[\frac{2}{3} u\sqrt{u} \right]_1^9 = \frac{2}{3} 9\sqrt{9} - \frac{2}{3} = \frac{52}{3}$$

< 24 ページ. 定積分の置換積分法 (1) >

問の解答

(1) $u = x^3 + 1$ とおくと

$$\int_{-1}^1 3x^2(x^3 + 1)^4 dx = \int_0^2 u^4 du = \left[\frac{1}{5} u^5 \right]_0^2 = \frac{32}{5}$$

(2) $u = x^2 + 1$ とおくと

$$\int_0^2 2x\sqrt{x^2 + 1} dx = \int_1^5 \sqrt{u} du = \left[\frac{2}{3} u\sqrt{u} \right]_1^5 = \frac{10\sqrt{5}}{3} - \frac{2}{3}$$

(3) $u = x^4 + 1$ とおくと

$$\int_0^1 \frac{4x^3}{(x^4 + 1)^2} dx = \int_1^2 \frac{1}{u^2} du = \left[-\frac{1}{u} \right]_1^2 = \frac{1}{2}$$

< 25 ページ. 定積分の置換積分法 (2) >

問の解答

(1) $u = x^2 + 2$ とおくと

$$\int_0^1 x(x^2 + 2)^3 dx = \int_2^3 \frac{1}{2} u^3 du = \left[\frac{1}{8} u^4 \right]_2^3 = \frac{1}{8} (81 - 16) = \frac{65}{8}$$

(2) $u = x^2$ とおくと

$$\int_0^3 x e^{x^2} dx = \int_0^9 \frac{1}{2} e^u du = \left[\frac{1}{2} e^u \right]_0^9 = \frac{e^9}{2} - \frac{1}{2}$$

(3) $u = x^3 + 2$ とおくと

$$\int_{-1}^2 \frac{x^2}{x^3 + 2} dx = \int_1^{10} \frac{1}{u} \times \frac{1}{3} du = \left[\frac{1}{3} \log u \right]_1^{10} = \frac{1}{3} \log 10$$

(4) $u = x^2 + 1$ とおくと

$$\int_0^2 \frac{x}{(x^2 + 1)^3} du = \int_1^5 \frac{1}{2u^3} du = \left[-\frac{1}{4u^2} \right]_1^5 = -\frac{1}{100} + \frac{1}{4} = \frac{6}{25}$$

< 26 ページ. 定積分の置換積分法 (3) >

問の解答

$$(1) \int_0^1 x(1-x)^7 dx = \int_1^0 (1-u)u^7(-1) du = \int_1^0 (u-1)u^7 du$$

$$\left(\begin{array}{l} u = 1 - x \\ \Downarrow \\ \frac{du}{dx} = -1 \\ x = 1 - u \end{array} \right) = \int_1^0 (u^8 - u^7) du = \left[\frac{u^9}{9} - \frac{u^8}{8} \right]_1^0$$

$$= 0 - \left(\frac{1}{9} - \frac{1}{8} \right) = \frac{1}{72}$$

$$(2) \int_2^5 x\sqrt{x-1} dx = \int_1^4 (u+1)\sqrt{u} du = \int_1^4 (u^{\frac{3}{2}} + u^{\frac{1}{2}}) du$$

$$\left(\begin{array}{l} u = x - 1 \\ \Downarrow \\ \frac{du}{dx} = 1 \\ x = u + 1 \end{array} \right) = \left[\frac{2}{5}x^{\frac{5}{2}} + \frac{2}{3}x^{\frac{3}{2}} \right]_1^4$$

$$= \left(\frac{2}{5} \times 4^{\frac{5}{2}} + \frac{2}{3} \times 4^{\frac{3}{2}} \right) - \left(\frac{2}{5} + \frac{2}{3} \right)$$

$$= \frac{64}{5} + \frac{16}{3} - \frac{2}{5} - \frac{2}{3} = \frac{62}{5} + \frac{14}{3}$$

$$= \frac{186 + 70}{15} = \frac{256}{15}$$

< 27 ページ. 定積分の置換積分法 (4) >

問の解答

$$(1) \int_{-1}^{\sqrt{3}} \sqrt{4-x^2} dx = \int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \sqrt{4-4\sin^2\theta} 2\cos\theta d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} 4\cos^2\theta d\theta$$

$$\begin{aligned} \left(\begin{array}{l} x = 2\sin\theta \\ \frac{dx}{d\theta} = 2\cos\theta \end{array} \right) &= \int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} (2 + 2\cos(2\theta)) d\theta = \left[2\theta + \sin(2\theta) \right]_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \\ &= \frac{2\pi}{3} + \sin\left(\frac{2\pi}{3}\right) - \left(-\frac{\pi}{3} + \sin\left(-\frac{\pi}{3}\right)\right) \\ &= \frac{3\pi}{3} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \pi + \sqrt{3} \end{aligned}$$

$$(2) \int_0^{\sqrt{2}} \frac{dx}{\sqrt{4-x^2}} dx = \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{4-4\sin^2\theta}} 2\cos\theta d\theta = \int_0^{\frac{\pi}{4}} d\theta$$

$$(x = 2\sin\theta) \quad = \left[\theta \right]_0^{\frac{\pi}{4}} = \frac{\pi}{4}$$

< 28 ページ. 定積分の部分積分法 (1) >

問の解答

$$\begin{aligned}(1) \int_{-1}^1 (x+1)(x-1)^3 dx &= \left[\frac{1}{4}(x+1)(x-1)^4 \right]_{-1}^1 - \int_{-1}^1 \frac{1}{4}(x-1)^4 dx \\ &= 0 - \left[\frac{1}{20}(x-1)^5 \right]_{-1}^1 \\ &= -\left(0 - \frac{-32}{20} \right) = -\frac{8}{5}\end{aligned}$$

$$\begin{aligned}(2) \int_0^{\frac{\pi}{2}} x \sin x dx &= \left[-x \cos x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (-\cos x) dx = 0 + \int_0^{\frac{\pi}{2}} \cos x dx \\ &= \left[\sin x \right]_0^{\frac{\pi}{2}} = 1\end{aligned}$$

$$\begin{aligned}(3) \int_0^1 x e^x dx &= \left[x e^x \right]_0^1 - \int_0^1 e^x dx \\ &= (1e^1 - 0) - \left[e^x \right]_0^1 \\ &= e - (e - 1) = 1\end{aligned}$$

< 29 ページ. 定積分の部分積分法 (2) >

問の解答

$$\begin{aligned}
 (1) \int_{\frac{1}{e}}^e x \log x \, dx &= \left[\frac{x^2}{2} \log x \right]_{\frac{1}{e}}^e - \int_{\frac{1}{e}}^e \frac{x}{2} \, dx \\
 &= \frac{e^2}{2} \log e - \frac{1}{2e^2} \log \left(\frac{1}{e} \right) - \left[\frac{x^2}{4} \right]_{\frac{1}{e}}^e \\
 &= \frac{e^2}{2} + \frac{1}{2e^2} - \left(\frac{e^2}{4} - \frac{1}{4e^2} \right) \\
 &= \frac{e^2}{4} + \frac{3}{4e^2}
 \end{aligned}$$

$$\begin{aligned}
 (2) \int_1^{\sqrt{e}} x^3 \log x \, dx &= \left[\frac{x^4}{4} \log x \right]_1^{\sqrt{e}} - \int_1^{\sqrt{e}} \frac{x^3}{4} \, dx \\
 &= \frac{e^2}{4} \log \sqrt{e} - 0 - \left[\frac{x^4}{16} \right]_1^{\sqrt{e}} \\
 &= \frac{e^2}{8} - \left(\frac{e^2}{16} - \frac{1}{16} \right) \\
 &= \frac{e^2}{16} + \frac{1}{16}
 \end{aligned}$$

$$\begin{aligned}
 (3) \int_0^1 x^2 e^{x+1} \, dx &= \left[x^2 e^{x+1} \right]_0^1 - \int_0^1 2x e^{x+1} \, dx \\
 &= (1e^2 - 0) - \left\{ \left[2x e^{x+1} \right]_0^1 - \int_0^1 2e^{x+1} \, dx \right\} \\
 &= e^2 - (2e^2 - 0) + \left[2e^{x+1} \right]_0^1 \\
 &= -e^2 + (2e^2 - 2e^1) = e^2 - 2e
 \end{aligned}$$

$$\begin{aligned}
 (4) \int_0^1 x^3 e^x \, dx &= \left[x^3 e^x \right]_0^1 - \int_0^1 3x^2 e^x \, dx \\
 &= (1e^1 - 0) - \left\{ \left[3x^2 e^x \right]_0^1 - \int_0^1 6x e^x \, dx \right\} \\
 &= e - (3e^1 - 0) + \left[6x e^x \right]_0^1 - \int_0^1 6e^x \, dx \\
 &= -2e + (6e^1 - 0) - \left[6e^x \right]_0^1 \\
 &= 4e - (6e^1 - 6) = 6 - 2e
 \end{aligned}$$

< 30 ページ. 定積分の部分積分法 (3) >

問の解答

$$\begin{aligned}
 (1) \int_0^{\frac{\pi}{2}} x^2 \cos x \, dx &= \left[x^2 \sin x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2x \sin x \, dx \\
 &= \left(\frac{\pi}{2} \right)^2 \sin \left(\frac{\pi}{2} \right) - 0 - \left\{ \left[2x \times (-\cos x) \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2 \times (-\cos x) \, dx \right\} \\
 &= \frac{\pi^2}{4} - 0 - \int_0^{\frac{\pi}{2}} 2 \cos x \, dx \\
 &= \frac{\pi^2}{4} - \left[2 \sin x \right]_0^{\frac{\pi}{2}} = \frac{\pi^2}{4} - 2
 \end{aligned}$$

$$\begin{aligned}
 (2) \int_0^{\frac{\pi}{2}} x^2 \sin(2x) \, dx &= \left[\frac{-x^2 \cos 2x}{2} \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (-x \cos(2x)) \, dx \\
 &= -\frac{\left(\frac{\pi}{2}\right)^2}{2} \cos \pi + \int_0^{\frac{\pi}{2}} x \cos(2x) \, dx \\
 &= \frac{\pi^2}{8} + \left[x \frac{\sin(2x)}{2} \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{\sin(2x)}{2} \, dx \\
 &= \frac{\pi^2}{8} + \frac{\pi}{2} \times \frac{\sin(\pi)}{2} - 0 + \left[\frac{\cos(2x)}{4} \right]_0^{\frac{\pi}{2}} \\
 &= \frac{\pi^2}{8} - \frac{1}{2}
 \end{aligned}$$

< 31 ページ. 定積分の練習 (1) >

問の解答

$$(1) \int_{-1}^3 dx = 3 + 1 = 4$$

$$(2) \int_1^{\sqrt{e}} \frac{dx}{x} = [\log x]_1^{\sqrt{e}} = \log \sqrt{e} - \log 1 = \frac{1}{2}$$

$$(3) \int_0^1 \sqrt[3]{x} dx = \left[\frac{3}{4} x^{\frac{4}{3}} \right]_0^1 = \frac{3}{4}$$

$$(4) \int_0^{\pi} (3 \sin x - 4 \cos x) dx = [-3 \cos x - 4 \sin x]_0^{\pi} \\ = -3 \cos \pi - 4 \sin \pi + 3 \cos 0 + 4 \sin 0 = 6$$

$$(5) \int_1^2 \frac{3x^2 - 4x + 1}{x^2} dx = \int_1^2 \left(3 - \frac{4}{x} + \frac{1}{x^2} \right) dx = \left[3x - 4 \log |x| - \frac{1}{x} \right]_1^2 \\ = \left(6 - 4 \log 2 - \frac{1}{2} \right) - \left(3 - 4 \log 1 - \frac{1}{1} \right) = \frac{7}{2} - 4 \log 2$$

$$(6) \int_1^9 \frac{dx}{\sqrt{x}} = [2\sqrt{x}]_1^9 = 2\sqrt{9} - 2 = 4$$

$$(7) \int_{-\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{dx}{\cos^2 x} = [\tan x]_{-\frac{\pi}{3}}^{\frac{\pi}{4}} = 1 + \sqrt{3}$$

$$(8) \int_1^2 \frac{1}{3x+1} dx = \left[\frac{1}{3} \log |3x+1| \right]_1^2 = \frac{1}{3} \log 7 - \frac{1}{3} \log 4 = \frac{1}{3} \log \left(\frac{7}{4} \right)$$

$$(9) \int_2^3 \frac{dx}{x^2-1} = \int_2^3 \frac{1}{2} \left(\frac{1}{x-1} - \frac{1}{x+1} \right) dx = \left[\frac{1}{2} \log \left| \frac{x-1}{x+1} \right| \right]_2^3 = \frac{1}{2} \log \left(\frac{3}{2} \right)$$

$$(10) \int_0^{\pi} \sin 2x \cos x dx = \int_0^{\pi} \frac{1}{2} \{ \sin(3x) + \sin x \} dx = \left[-\frac{1}{6} \cos(3x) - \frac{1}{2} \cos x \right]_0^{\pi} \\ = \left(-\frac{1}{6} \cos(3\pi) - \frac{1}{2} \cos \pi \right) + \left(\frac{1}{6} \cos 0 + \frac{1}{2} \cos 0 \right) = \frac{4}{3}$$

$$(11) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x dx = 2 \int_0^{\frac{\pi}{2}} \frac{1}{2} (1 - \cos(2x)) dx = \left[x - \frac{1}{2} \sin(2x) \right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$$(12) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(2x) dx = 2 \int_0^{\frac{\pi}{2}} \frac{1}{2} (1 + \cos(4x)) dx = \left[x + \frac{1}{4} \sin(4x) \right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$$(13) \int_{-2}^2 e^{3x-1} dx = \left[\frac{1}{3} e^{3x-1} \right]_{-2}^2 = \frac{1}{3} e^5 - \frac{1}{3} e^{-7} = \frac{1}{3} \left(e^5 - \frac{1}{e^7} \right)$$

$$(14) \int_{-1}^1 x e^{-x^2} dx = \left[-\frac{1}{2} e^{-x^2} \right]_{-1}^1 = -\frac{1}{2} e^{-1} + \frac{1}{2} e^{-1} = 0$$

< 32 ページ. 定積分の練習 (2) >

問の解答

$$(1) \int_0^1 \frac{1}{(3x+1)^5} dx = \int_1^4 \frac{1}{3} \times \frac{1}{u^5} du = \left[-\frac{1}{12} u^{-4} \right]_1^4 = -\frac{1}{12} \left\{ \frac{1}{4^4} - \frac{1}{1^4} \right\} = \frac{85}{1024}$$

$$(2) \int_1^{10} \sqrt{5x-1} dx = \int_4^{49} \frac{1}{5} \sqrt{u} du = \left[\frac{2}{15} u^{\frac{3}{2}} \right]_4^{49} = \frac{2}{15} \left\{ 49^{\frac{3}{2}} - 4^{\frac{3}{2}} \right\} = \frac{2}{15} \{ 343 - 8 \} = \frac{134}{3}$$

$$(3) \int_0^1 \frac{x^2}{(x^3+1)^4} dx = \int_1^2 \frac{1}{3} \times \frac{1}{u^4} du = \left[-\frac{1}{9} u^{-3} \right]_1^2 = -\frac{1}{9} \left\{ \frac{1}{2^3} - \frac{1}{1^3} \right\} = \frac{1}{9} \times \frac{7}{8} = \frac{7}{72}$$

$$(4) \int_0^1 \frac{3x}{x^2+1} dx = \left[\frac{3}{2} \log(x^2+1) \right]_0^1 = \frac{3}{2} \log 2$$

$$(5) \int_{-\sqrt{3}}^2 \sqrt{4-x^2} dx = \int_{-\frac{\pi}{3}}^{\frac{\pi}{2}} \sqrt{4-4\sin^2\theta} 2\cos\theta d\theta = \int_{-\frac{\pi}{3}}^{\frac{\pi}{2}} 4\cos^2\theta d\theta$$

$$\begin{aligned} (x=2\sin\theta) \quad &= \int_{-\frac{\pi}{3}}^{\frac{\pi}{2}} \{2+2\cos(2\theta)\} d\theta = \left[2\theta + \sin(2\theta) \right]_{-\frac{\pi}{3}}^{\frac{\pi}{2}} \\ &= \pi + \sin\pi - \left(-\frac{2\pi}{3} + \sin\left(-\frac{2\pi}{3}\right) \right) = \frac{5\pi}{3} + \frac{\sqrt{3}}{2} \end{aligned}$$

$$\begin{aligned} (6) \int_0^1 x(x+1)^4 dx &= \left[x \frac{(x+1)^5}{5} \right]_0^1 - \int_0^1 \frac{(x+1)^5}{5} dx = \frac{2^5}{5} - 0 - \left[\frac{(x+1)^6}{30} \right]_0^1 \\ &= \frac{32}{5} - \left(\frac{2^6}{30} - \frac{1^6}{30} \right) = \frac{43}{10} \end{aligned}$$

$$\begin{aligned} (7) \int_{-\frac{\pi}{4}}^{\frac{\pi}{3}} x \sin x dx &= \left[-x \cos x \right]_{-\frac{\pi}{4}}^{\frac{\pi}{3}} - \int_{-\frac{\pi}{4}}^{\frac{\pi}{3}} (-\cos x) dx = -\frac{\pi}{3} \cos \frac{\pi}{3} + \frac{\pi}{4} \cos\left(-\frac{\pi}{4}\right) + \left[\sin x \right]_{-\frac{\pi}{4}}^{\frac{\pi}{3}} \\ &= -\frac{\pi}{6} + \frac{\sqrt{2}}{8}\pi + \sin \frac{\pi}{3} - \sin\left(-\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{8} - \frac{1}{6} \right)\pi + \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \end{aligned}$$

$$(8) \int_{\frac{1}{e}}^1 \log x dx = \left[x \log x - x \right]_{\frac{1}{e}}^1 = (1 \log 1 - 1) - \left(\frac{1}{e} \log \frac{1}{e} - \frac{1}{e} \right) = \frac{2}{e} - 1$$

$$\begin{aligned} (9) \int_{-1}^0 x^2 e^x dx &= \left[x^2 e^x \right]_{-1}^0 - \int_{-1}^0 2x e^x dx = (0 - e^{-1}) - \left\{ \left[2x e^x \right]_{-1}^0 - \int_{-1}^0 2e^x dx \right\} \\ &= -\frac{1}{e} - (0 + 2e^{-1}) + \left[2e^x \right]_{-1}^0 = -\frac{3}{e} + 2e^0 - 2e^{-1} = 2 - \frac{5}{e} \end{aligned}$$

$$\begin{aligned} (10) \int_0^\pi x^2 \cos x dx &= \left[x^2 \sin x \right]_0^\pi - \int_0^\pi 2x \sin x dx = 0 - 0 + \int_0^\pi 2x(\cos x)' dx \\ &= \left[2x \cos x \right]_0^\pi - \int_0^\pi 2 \cos x dx = 2\pi \cos \pi - 0 - \left[2 \sin x \right]_0^\pi = -2\pi \end{aligned}$$

< 33 ページ. 和の極限と定積分 (1) >

問の解答

$$(1) \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{1}{\left(2 + \frac{k}{n}\right)^2} = \int_2^3 \frac{1}{x^2} dx = \left[-\frac{1}{x}\right]_2^3 = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6}$$

$$(2) \lim_{n \rightarrow \infty} \frac{2}{n} \sum_{k=1}^n \left(1 + \frac{2k}{n}\right)^3 = \int_1^3 x^3 dx = \left[\frac{1}{4}x^4\right]_1^3 = \frac{81}{4} - \frac{1}{4} = \frac{80}{4} = 20$$

$$(3) \lim_{n \rightarrow \infty} \frac{5}{n} \sum_{k=1}^n \sqrt{4 + \frac{5k}{n}} = \int_4^9 \sqrt{x} dx = \left[\frac{2}{3}x\sqrt{x}\right]_4^9 = \frac{2}{3}(9\sqrt{9} - 4\sqrt{4}) = \frac{38}{3}$$

$$(4) \lim_{n \rightarrow \infty} \frac{3}{n} \sum_{k=1}^n \frac{1}{\sqrt{1 + \frac{3k}{n}}} = \int_1^4 \frac{1}{\sqrt{x}} dx = \left[2\sqrt{x}\right]_1^4 = 2\sqrt{4} - 2 = 2$$

< 34 ページ. 和の極限と定積分 (2) >

問の解答

$$\begin{aligned}
 (1) \quad & \lim_{n \rightarrow \infty} \frac{3}{n} \left\{ \sqrt{1 + \frac{3}{n}} + \sqrt{1 + \frac{6}{n}} + \sqrt{1 + \frac{9}{n}} + \cdots + \sqrt{1 + \frac{3n}{n}} \right\} \\
 &= \lim_{n \rightarrow \infty} \frac{3}{n} \sum_{k=1}^n \sqrt{1 + \frac{3}{n}k} = \int_1^4 \sqrt{x} \, dx = \left[\frac{2}{3} x \sqrt{x} \right]_1^4 = \frac{2}{3} (4\sqrt{4} - 1) = \frac{14}{3}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & \lim_{n \rightarrow \infty} \frac{\pi}{n} \left\{ \cos\left(-\frac{\pi}{2} + \frac{\pi}{n}\right) + \sin\left(-\frac{\pi}{2} + \frac{2\pi}{n}\right) + \cos\left(-\frac{\pi}{2} + \frac{3\pi}{n}\right) + \cdots + \cos\left(-\frac{\pi}{2} + \frac{n\pi}{n}\right) \right\} \\
 &= \lim_{n \rightarrow \infty} \frac{\pi}{n} \sum_{k=1}^n \cos\left(-\frac{\pi}{2} + \frac{k\pi}{n}\right) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x) \, dx = \left[\sin x \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & \lim_{n \rightarrow \infty} \frac{5}{n} \left\{ \frac{1}{\sqrt{4 + \frac{5}{n}}} + \frac{1}{\sqrt{4 + \frac{10}{n}}} + \frac{1}{\sqrt{4 + \frac{15}{n}}} + \cdots + \frac{1}{\sqrt{4 + \frac{5n}{n}}} \right\} \\
 &= \lim_{n \rightarrow \infty} \frac{5}{n} \sum_{k=1}^n \frac{1}{\sqrt{4 + \frac{5}{n}k}} = \int_4^9 \frac{1}{\sqrt{x}} \, dx = \left[2\sqrt{x} \right]_4^9 = 2\sqrt{9} - 2\sqrt{4} = 6 - 4 = 2
 \end{aligned}$$

< 35 ページ. 和の極限と定積分 (3) >

問の解答

$$(1) \lim_{n \rightarrow \infty} \frac{2}{n} \left\{ \left(-1 + \frac{2}{n}\right)^4 + \left(-1 + \frac{4}{n}\right)^4 + \left(-1 + \frac{6}{n}\right)^4 + \cdots + \left(-1 + \frac{2n}{n}\right)^4 \right\}$$

$$= \lim_{n \rightarrow \infty} \frac{2}{n} \sum_{k=1}^n \left(-1 + \frac{2k}{n}\right)^4 = \int_{-1}^1 x^4 dx = 2 \int_0^1 x^4 dx = 2 \left[\frac{1}{5} x^5 \right]_0^1 = \frac{2}{5}$$

$$(2) \lim_{n \rightarrow \infty} \frac{3}{n} \left\{ \frac{1}{\left(1 + \frac{3}{n}\right)^3} + \frac{1}{\left(1 + \frac{6}{n}\right)^3} + \frac{1}{\left(1 + \frac{9}{n}\right)^3} + \cdots + \frac{1}{\left(1 + \frac{3n}{n}\right)^3} \right\}$$

$$= \lim_{n \rightarrow \infty} \frac{3}{n} \sum_{k=1}^n \frac{1}{\left(1 + \frac{3k}{n}\right)^3} = \int_1^4 \frac{1}{x^3} dx = \left[-\frac{1}{2x^2} \right]_1^4 = -\frac{1}{32} + \frac{1}{2} = \frac{15}{32}$$

$$(3) \lim_{n \rightarrow \infty} \frac{1}{n} \left\{ \frac{1}{1 + \frac{1}{n}} + \frac{1}{1 + \frac{2}{n}} + \frac{1}{1 + \frac{3}{n}} + \cdots + \frac{1}{1 + \frac{n}{n}} \right\}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{1}{1 + \frac{k}{n}} = \int_1^2 \frac{1}{x} dx = \left[\log x \right]_1^2 = \log 2$$

$$(4) \lim_{n \rightarrow \infty} \frac{1}{n} \left\{ \left(4 + \frac{1}{n}\right) + \left(4 + \frac{2}{n}\right) + \left(4 + \frac{3}{n}\right) + \cdots + \left(4 + \frac{n}{n}\right) \right\}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \left(4 + \frac{k}{n}\right) = \int_4^5 x dx = \left[\frac{x^2}{2} \right]_4^5 = \frac{1}{2}(25 - 16) = \frac{9}{2}$$

$$(5) \lim_{n \rightarrow \infty} \frac{4}{n} \left\{ \sqrt{\frac{4}{n}} + \sqrt{\frac{8}{n}} + \sqrt{\frac{12}{n}} + \cdots + \sqrt{\frac{4n}{n}} \right\}$$

$$= \lim_{n \rightarrow \infty} \frac{4}{n} \sum_{k=1}^n \sqrt{\frac{4k}{n}} = \int_0^4 \sqrt{x} dx = \left[\frac{2}{3} x \sqrt{x} \right]_0^4 = \frac{2}{3} 4 \sqrt{4} = \frac{16}{3}$$

$$(6) \lim_{n \rightarrow \infty} \frac{\pi}{n} \left\{ \sin\left(\frac{\pi}{n}\right) + \sin\left(\frac{2\pi}{n}\right) + \sin\left(\frac{3\pi}{n}\right) + \cdots + \sin\left(\frac{n\pi}{n}\right) \right\}$$

$$= \lim_{n \rightarrow \infty} \frac{\pi}{n} \sum_{k=1}^n \sin\left(\frac{\pi k}{n}\right) = \int_0^\pi \sin x dx = \left[-\cos x \right]_0^\pi = 2$$

$$(7) \lim_{n \rightarrow \infty} \frac{\pi}{2n} \left\{ \cos\left(\frac{\pi}{2n}\right) + \cos\left(\frac{2\pi}{2n}\right) + \cos\left(\frac{3\pi}{2n}\right) + \cdots + \cos\left(\frac{n\pi}{2n}\right) \right\}$$

$$= \lim_{n \rightarrow \infty} \frac{\pi}{2n} \sum_{k=1}^n \cos\left(\frac{\pi k}{2n}\right) = \int_0^{\frac{\pi}{2}} \cos x dx = \left[\sin x \right]_0^{\frac{\pi}{2}} = 1$$

$$(8) \lim_{n \rightarrow \infty} \frac{1}{n} \left\{ \frac{1}{1 + \left(\frac{1}{n}\right)^2} + \frac{1}{1 + \left(\frac{2}{n}\right)^2} + \frac{1}{1 + \left(\frac{3}{n}\right)^2} + \cdots + \frac{1}{1 + \left(\frac{n}{n}\right)^2} \right\}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{1}{1 + \left(\frac{k}{n}\right)^2} = \int_0^1 \frac{1}{1 + x^2} dx = \left[\tan^{-1} x \right]_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4}$$

$$(9) \lim_{n \rightarrow \infty} \frac{1}{n} \left\{ \frac{\frac{1}{n}}{1 + \left(\frac{1}{n}\right)^2} + \frac{\frac{2}{n}}{1 + \left(\frac{2}{n}\right)^2} + \frac{\frac{3}{n}}{1 + \left(\frac{3}{n}\right)^2} + \cdots + \frac{\frac{n}{n}}{1 + \left(\frac{n}{n}\right)^2} \right\}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{\frac{k}{n}}{1 + \left(\frac{k}{n}\right)^2} = \int_0^1 \frac{x}{1 + x^2} dx = \left[\frac{1}{2} \log(1 + x^2) \right]_0^1 = \frac{1}{2} \log 2$$

< 36 ページ. 面積 (1) >

問の解答

$$(1) \int_0^1 e^x dx = e - 1$$

$$(2) \int_1^9 \sqrt{x} dx = \left[\frac{2}{3} x \sqrt{x} \right]_1^9 = \frac{2}{3} (9\sqrt{9} - 1) = \frac{52}{3}$$

$$(3) \int_1^2 \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}$$

$$(4) \int_1^2 \frac{1}{x} dx = \left[\log x \right]_1^2 = \log 2$$

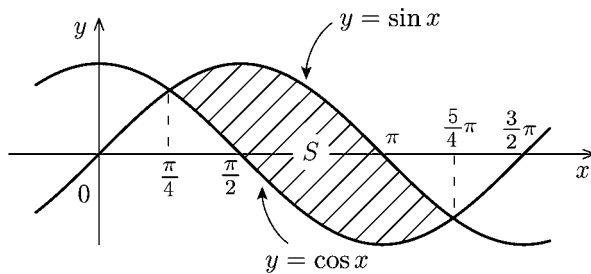
< 37 ページ. 面積 (2) >

問1の解答

$$S = \int_a^b \{0 - g(x)\} dx = - \int_a^b g(x) dx$$

問2の解答

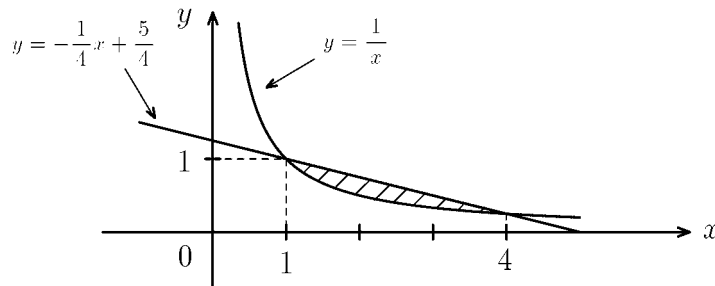
$$\begin{aligned} S &= \int_{\frac{\pi}{4}}^{\frac{5}{4}\pi} (\sin x - \cos x) dx = \left[-\cos x - \sin x \right]_{\frac{\pi}{4}}^{\frac{5}{4}\pi} \\ &= -\cos\left(\frac{5}{4}\pi\right) - \sin\left(\frac{5}{4}\pi\right) + \cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right) = 2\sqrt{2} \end{aligned}$$



問3の解答

$$(1) \int_0^1 (\sqrt{x} - x^2) dx = \left[\frac{2}{3} x\sqrt{x} - \frac{x^3}{3} \right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

$$\begin{aligned} (2) \int_1^4 \left(-\frac{1}{4}x + \frac{5}{4} - \frac{1}{x} \right) dx &= \left[-\frac{x^2}{8} + \frac{5}{4}x - \log x \right]_1^4 \\ &= -\frac{16}{8} + 5 - \log 4 - \left(-\frac{1}{8} + \frac{5}{4} \right) = 3 + \frac{1}{8} - \frac{5}{4} - \log 4 = \frac{15}{8} - \log 4 \end{aligned}$$



< 38 ページ. 面積 (3) >

問の解答

$$\begin{aligned} S &= \int_0^{\frac{\sqrt{3}}{2}a} \sqrt{a^2 - x^2} dx = \int_0^{\frac{\pi}{3}} \sqrt{a^2 - a^2 \sin^2 \theta} a \cos \theta d\theta && \left(\begin{array}{l} x = a \sin \theta \\ x \mid 0 \rightarrow \frac{\sqrt{3}}{2}a \\ \theta \mid 0 \rightarrow \frac{\pi}{3} \end{array} \right) \\ &= \int_0^{\frac{\pi}{3}} a^2 \cos^2 \theta d\theta = a^2 \int_0^{\frac{\pi}{3}} \frac{1 + \cos(2\theta)}{2} d\theta \\ &= a^2 \left[\frac{\theta}{2} + \frac{1}{4} \sin(2\theta) \right]_0^{\frac{\pi}{3}} = a^2 \left\{ \frac{\pi}{6} + \frac{1}{4} \sin \left(\frac{2\pi}{3} \right) - 0 \right\} \\ &= \frac{a^2 \pi}{6} + \frac{\sqrt{3}}{8} a^2 \end{aligned}$$

< 39 ページ. 面積 (4) >

問 1 の解答

$$\begin{aligned}
 (1) \int_0^r \sqrt{r^2 - x^2} dx &= \int_0^{\frac{\pi}{2}} \sqrt{r^2 - r^2 \sin^2 \theta} r \cos \theta d\theta \\
 &= \int_0^{\frac{\pi}{2}} r^2 \cos^2 \theta d\theta = \int_0^{\frac{\pi}{2}} \frac{r^2}{2} \{1 + \cos(2\theta)\} d\theta \\
 &= \left[\frac{r^2}{2} \theta + \frac{r^2}{4} \sin(2\theta) \right]_0^{\frac{\pi}{2}} = \frac{\pi r^2}{4}
 \end{aligned}$$

$$(2) S = 4 \times \frac{S}{4} = 4 \times \frac{\pi r^2}{4} = \pi r^2$$

問 2 の解答

$$(1) y = \sqrt{b^2 - \left(\frac{b}{a}x\right)^2} = \frac{b}{a} \sqrt{a^2 - x^2} dx$$

$$(2) \frac{S}{4} = \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} dx$$

(3) $x = a \sin \theta$ とおくと

$$\begin{aligned}
 \frac{S}{4} &= \int_0^{\frac{\pi}{2}} \frac{b}{a} \sqrt{a^2 - a^2 \sin^2 \theta} a \cos \theta d\theta = \int_0^{\frac{\pi}{2}} ab \cos^2 \theta d\theta \\
 &= \int_0^{\frac{\pi}{2}} \frac{ab}{2} \{1 + \cos(2\theta)\} d\theta = \left[\frac{ab}{2} \theta + \frac{ab}{4} \sin(2\theta) \right]_0^{\frac{\pi}{2}} \\
 &= \frac{ab}{4} \pi
 \end{aligned}$$

$$\text{よって } S = 4 \times \frac{S}{4} = 4 \times \frac{ab\pi}{4} = \pi ab$$

< 40 ページ. 体積 (1) >

問 1 の解答

$$S_1 = \int_{-1}^1 \{(4 - x^2) - (1 - x^2)\} dx = \int_{-1}^1 3dx = [3x]_{-1}^1 = 6$$

S_2 は長方形の面積だから

$$S_2 = \text{底辺} \times \text{高さ} = 2 \times 3 = 6$$

問 2 の解答

(1) $S(x) = S$

(2) $V = Sh$

< 41 ページ. 体積 (2) >

問の解答

$$(1) \quad A'C' = \frac{ax}{h} \quad , \quad B'C' = \frac{bx}{h}$$

$$(2) \quad S(x) = \frac{1}{2}A'C' \times B'C' = \frac{abx^2}{2h^2}$$

$$(3) \quad V = \int_0^h S(x)dx = \int_0^h \frac{abx^2}{2h^2}dx = \left[\frac{abx^3}{6h^2} \right]_0^h = \frac{abh}{6}$$

< 42 ページ. 体積 (3) >

問 1 の解答

$$S(x) = \left(\frac{x}{2}\right)^2$$

$$V = \int_0^4 S(x) dx = \int_0^4 \frac{x^2}{4} dx = \left[\frac{x^3}{12}\right]_0^4 = \frac{4^3}{12} = \frac{16}{3}$$

問 2 の解答

$$(1) \quad S(x) = \left(\frac{ax}{h}\right) \times \left(\frac{bx}{h}\right) = \frac{abx^2}{h^2}$$

$$V = \int_0^h S(x) dx = \int_0^h \frac{abx^2}{h^2} dx = \left[\frac{abx^3}{3h^2}\right]_0^h = \frac{abh}{3}$$

$$(2) \quad S = ab \text{ より}$$

$$\frac{S(x)}{S} = \frac{\frac{abx^2}{h^2}}{ab} = \frac{x^2}{h^2}$$

問 3 の解答

$$(1) \quad S(x) = \pi \times \left(\frac{rx}{h}\right)^2 = \frac{\pi r^2 x^2}{h^2}$$

$$(2) \quad V = \int_0^h S(x) dx = \int_0^h \frac{\pi r^2 x^2}{h^2} dx = \left[\frac{\pi r^2 x^3}{3h^2}\right]_0^h = \frac{\pi r^2 h}{3}$$

$$(3) \quad S = \pi r^2 \text{ より}$$

$$\frac{S(x)}{S} = \frac{\frac{\pi r^2 x^2}{h^2}}{\pi r^2} = \frac{x^2}{h^2}$$

< 43 ページ. 体積 (4) >

問 1 の解答

$$(1) S(x) = \pi \{f(x)\}^2$$

$$(2) V = \int_a^b \pi \{f(x)\}^2 dx$$

問 2 の解答

$$V = \int_0^h \pi \left(\frac{r}{h}x\right)^2 dx = \int_0^h \frac{\pi r^2 x^2}{h^2} dx = \left[\frac{\pi r^2 x^3}{3h^2}\right]_0^h = \frac{\pi r^2 h}{3}$$

問 3 の解答

$$\begin{aligned} V &= \int_{-r}^r \pi \left\{\sqrt{r^2 - x^2}\right\}^2 dx = \int_{-r}^r \pi(r^2 - x^2) dx = \left[\pi r^2 x - \frac{\pi}{3}x^3\right]_{-r}^r \\ &= \left(\pi r^3 - \frac{\pi}{3}r^3\right) - \left(-\pi r^3 + \frac{\pi}{3}r^3\right) = \frac{4}{3}\pi r^3 \end{aligned}$$

< 44 ページ. 体積 (5) >

問 1 の解答

$$(1) A_1' C_1' = \frac{5}{7}x$$

$$(2) A_2' C_2' = \frac{5}{7}x$$

$$(3) S_1(x) = \frac{1}{2} \left(\frac{5}{7}x \right)^2 = \frac{25}{98}x^2$$

$$(4) S_2(x) = \frac{25}{98}x^2$$

$$(5) V_1 = \int_0^7 S_1(x)dx = \int_0^7 \frac{25}{98}x^2 = \left[\frac{25}{3 \times 98}x^3 \right]_0^7 = \frac{175}{6}$$

$$V_2 = \int_0^7 S_2(x)dx = \int_0^7 \frac{25}{98}x^2 = \frac{175}{6}$$

問 2 の解答

$$(1) S(x) = \left(\frac{x}{h} \right)^2 S$$

$$(2) V = \int_0^h S(x)dx = \int_0^h \frac{S}{h^2}x^2 dx = \left[\frac{S}{h^2} \times \frac{x^3}{3} \right]_0^h = \frac{Sh}{3}$$

< 45 ページ. 平面上の運動 >

問の解答

(1) 速度 : $\boldsymbol{v}(t) = (6t^2, 6t)$

速さ : $|\boldsymbol{v}(t)| = \sqrt{(6t^2)^2 + (6t)^2} = 6t\sqrt{t^2 + 1}$

(2) 速度 : $\boldsymbol{v}(t) = (-2\sin t, 2\cos t)$

速さ : $|\boldsymbol{v}(t)| = \sqrt{(-2\sin t)^2 + (2\cos t)^2} = 2$

(3) 速度 : $\boldsymbol{v}(t) = (-3\sin t \cos^2 t, 3\cos t \sin^2 t)$

速さ : $|\boldsymbol{v}(t)| = \sqrt{(-3\sin t \cos^2 t)^2 + (3\cos t \sin^2 t)^2} = 3|\sin t \cos t|$

(4) 速度 : $\boldsymbol{v}(t) = (-e^{-t}\cos t - e^{-t}\sin t, -e^{-t}\sin t + e^{-t}\cos t)$

速さ : $|\boldsymbol{v}(t)| = \sqrt{(-e^{-t}\cos t - e^{-t}\sin t)^2 + (-e^{-t}\sin t + e^{-t}\cos t)^2}$
 $= e^{-t}\sqrt{2\cos^2 t + 2\sin^2 t} = \sqrt{2}e^{-t}$

< 47 ページ. 道のり (2) >

問 1 の解答

$$(1) (x(t), y(t)) = (\sin t, 0)$$

$$\begin{aligned}(2) \int_0^\pi |v(t)| dt &= \int_0^\pi |\cos t| dt \\ &= \int_0^{\frac{\pi}{2}} \cos t dt + \int_{\frac{\pi}{2}}^\pi (-\cos t) dt \\ &= \left[\sin t \right]_0^{\frac{\pi}{2}} + \left[-\sin t \right]_{\frac{\pi}{2}}^\pi = 2\end{aligned}$$

問 2 の解答

$$(1) (x(t), y(t)) = (\cos t, \sin t)$$

$$\begin{aligned}(2) \int_0^{2\pi} |v(t)| dt &= \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2} dt \\ &= \int_0^{2\pi} 1 dt = 2\pi\end{aligned}$$

< 48 ページ. 道のり (3) >

問の解答

$$\begin{aligned} (1) \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= (-e^{-t} \cos t - e^{-t} \sin t)^2 + (-e^{-t} \sin t + e^{-t} \cos t)^2 \\ &= (e^{-t})^2 \{2 \cos^2 t + 2 \sin^2 t\} = 2(e^{-t})^2 \end{aligned}$$

$$\begin{aligned} \ell &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^{2\pi} \sqrt{2(e^{-t})^2} dt \\ &= \int_0^{2\pi} \sqrt{2} e^{-t} dt = \left[-\sqrt{2} e^{-t}\right]_0^{2\pi} \\ &= -\sqrt{2} e^{-2\pi} + \sqrt{2} e^0 = \sqrt{2} \left(1 - \frac{1}{e^{2\pi}}\right) \end{aligned}$$

$$\begin{aligned} (2) \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= (-3 \sin t \cos^2 t)^2 + (3 \cos t \sin^2 t)^2 \\ &= 9 \sin^2 t \cos^2 t \{\cos^2 t + \sin^2 t\} \\ &= 9 \sin^2 t \cos^2 t \end{aligned}$$

$$\begin{aligned} \ell &= \int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^{\frac{\pi}{2}} 3 |\sin t \cos t| dt \\ &= \int_0^{\frac{\pi}{2}} \frac{3}{2} \sin 2t dt = \left[-\frac{3}{4} \cos 2t\right]_0^{\frac{\pi}{2}} \\ &= -\frac{3}{4} \cos \pi + \frac{3}{4} \cos 0 = \frac{3}{2} \end{aligned}$$

< 49 ページ. 定積分の応用 (1) >

問 1 の解答

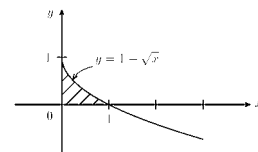
$$(1) \lim_{n \rightarrow \infty} \frac{1}{n} \left(\sqrt{1 + \frac{1}{n}} + \sqrt{1 + \frac{2}{n}} + \cdots + \sqrt{1 + \frac{n}{n}} \right) = \int_1^2 \sqrt{x} \, dx = \left[\frac{2}{3} x \sqrt{x} \right]_1^2 = \frac{2}{3} (2\sqrt{2} - 1)$$

$$(2) \lim_{n \rightarrow \infty} \frac{4}{n} \left(\frac{1}{\sqrt{\frac{4}{n}}} + \frac{1}{\sqrt{\frac{8}{n}}} + \cdots + \frac{1}{\sqrt{\frac{4n}{n}}} \right) = \int_0^4 \frac{1}{\sqrt{x}} \, dx = \left[2\sqrt{x} \right]_0^4 = 4$$

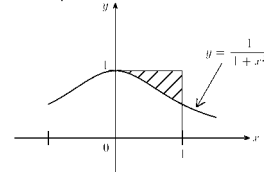
$$(3) \lim_{n \rightarrow \infty} \frac{\pi}{2n} \left(\sin\left(\frac{\pi}{2n}\right) + \sin\left(\frac{2\pi}{2n}\right) + \cdots + \sin\left(\frac{n\pi}{2n}\right) \right) = \int_0^{\frac{\pi}{2}} \sin(x) \, dx = \left[-\cos x \right]_0^{\frac{\pi}{2}} = 1$$

問 2 の解答

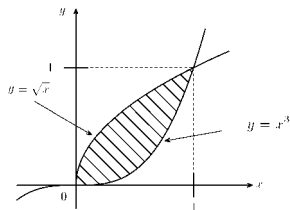
$$(1) \int_0^1 (1 - \sqrt{x}) \, dx = \left[x - \frac{2}{3} x \sqrt{x} \right]_0^1 = 1 - \frac{2}{3} = \frac{1}{3}$$



$$(2) \int_0^1 \left(1 - \frac{1}{1+x^2} \right) dx = \left[x - \tan^{-1}(x) \right]_0^1 = (1 - \tan^{-1} 1) - (0 - \tan^{-1} 0) = 1 - \frac{\pi}{4}$$



$$(3) \int_0^1 (\sqrt{x} - x^3) \, dx = \left[\frac{2}{3} x \sqrt{x} - \frac{1}{4} x^4 \right]_0^1 = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$$

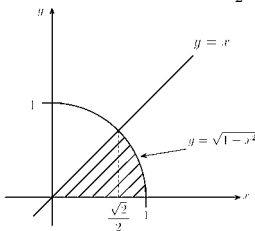


問 3 の解答

$$(1) \int_0^{\frac{\sqrt{2}}{2}} \pi x^2 \, dx + \int_{\frac{\sqrt{2}}{2}}^1 \pi (\sqrt{1-x^2})^2 \, dx = \left[\frac{\pi x^3}{3} \right]_0^{\frac{\sqrt{2}}{2}} + \pi \left[x - \frac{x^3}{3} \right]_{\frac{\sqrt{2}}{2}}^1$$

$$= \frac{\pi}{3 \times 2\sqrt{2}} + \pi \left\{ \left(1 - \frac{1}{3} \right) - \left(\frac{1}{\sqrt{2}} - \frac{1}{3 \times 2\sqrt{2}} \right) \right\}$$

$$= \pi \left\{ \frac{1}{3\sqrt{2}} + \frac{2}{3} - \frac{1}{\sqrt{2}} \right\} = \left(\frac{2 - \sqrt{2}}{3} \right) \pi$$



$$(2) \int_{-a}^a \pi \left\{ \frac{b}{a} \sqrt{a^2 - x^2} \right\}^2 dx = \frac{2\pi b^2}{a^2} \int_0^a (a^2 - x^2) dx = \frac{2\pi b^2}{a^2} \left[a^2 x - \frac{x^3}{3} \right]_0^a = \frac{2\pi b^2}{a^2} \times \frac{2a^3}{3}$$

$$= \frac{4\pi}{3} ab^2$$

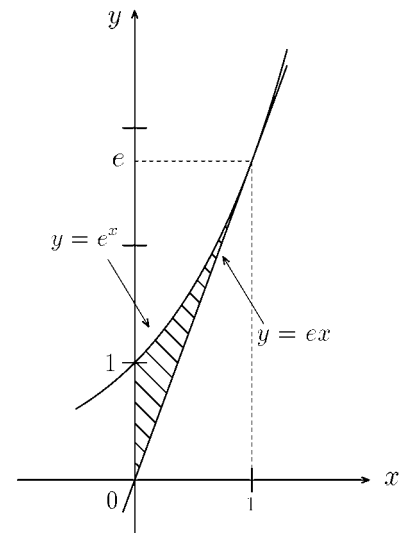
問 4 の解答

$$S = \int_0^1 (e^x - ex) \, dx = \left[e^x - \frac{e}{2} x^2 \right]_0^1$$

$$= \left(e^1 - \frac{e}{2} \right) - (e^0 - 0) = \frac{e}{2} - 1$$

$$V = \int_0^1 \pi (e^x)^2 \, dx - \int_0^1 \pi (ex)^2 \, dx$$

$$= \pi \left\{ \left[\frac{1}{2} e^{2x} \right]_0^1 - \left[\frac{e^2}{3} x^3 \right]_0^1 \right\} = \left(\frac{e^2}{6} - \frac{1}{2} \right) \pi$$



< 50 ページ. 定積分の応用 (2) >

問1の解答

$$(1) \left(x(t), y(t) \right) = (1 - \cos t, 0)$$

$$\begin{aligned} (2) \int_0^{2\pi} |\mathbf{v}(t)| dt &= \int_0^{2\pi} |\sin t| dt = \int_0^{\pi} \sin t dt + \int_{\pi}^{2\pi} (-\sin t) dt \\ &= \left[-\cos t \right]_0^{\pi} + \left[\cos t \right]_{\pi}^{2\pi} \\ &= -\cos \pi + \cos 0 + \cos 2\pi - \cos \pi = 4 \end{aligned}$$

問2の解答

$$(1) \left(x(t), y(t) \right) = \left(t, \frac{2}{3}t\sqrt{t} \right)$$

$$\begin{aligned} (2) \int_0^3 |\mathbf{v}(t)| dt &= \int_0^3 \sqrt{1^2 + (\sqrt{t})^2} dt = \int_0^3 \sqrt{1+t} dt = \int_0^3 (1+t)^{\frac{1}{2}} dt \\ &= \left[\frac{2}{3}(1+t)^{\frac{3}{2}} \right]_0^3 = \frac{2}{3} \left\{ 4^{\frac{3}{2}} - 1^{\frac{3}{2}} \right\} = \frac{14}{3} \end{aligned}$$

問3の解答

$$\begin{aligned} (1) \ell &= \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^1 \sqrt{(6t^2)^2 + (6t)^2} dt \\ &= \int_0^1 6t\sqrt{t^2+1} dt = \left[2(t^2+1)^{\frac{3}{2}} \right]_0^1 = 2 \times 2^{\frac{3}{2}} - 2 = 4\sqrt{2} - 2 \end{aligned}$$

$$\begin{aligned} (2) \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= (1 - \cos t)^2 + (\sin t)^2 = 1 - 2\cos t + \cos^2 t + \sin^2 t \\ &= 2 - 2\cos t = 4\sin^2\left(\frac{t}{2}\right) \end{aligned}$$

$$\begin{aligned} \ell &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^{2\pi} \sqrt{4\sin^2\left(\frac{t}{2}\right)} dt \\ &= \int_0^{2\pi} 2\sin\left(\frac{t}{2}\right) dt = \left[-4\cos\left(\frac{t}{2}\right) \right]_0^{2\pi} = -4\cos \pi + 4\cos 0 = 8 \end{aligned}$$